Lecture 1 - Introduction to Probability

6.042 - February 4, 2003

Let's play *Carnival Dice*! You roll three dice, one colored red, one green, and one blue. If the number six comes up on at least one of the dice, I pay you a dollar. Otherwise, you pay me a dollar. This is a perfectly fair game. After all, if you roll only one die, then the probability that a six comes up is 1 in 6. So if you roll *three* dice, then the probability that a six comes up is obviously:

$$3 \cdot \frac{1}{6} = \frac{1}{2}$$

Therefore, you have an even chance of winning! Or do you?

This game teaches two lessons. First, *do not rely on intuition to solve probability problems*. For some reason, we all have an indomitable belief in our ability to figure "the odds". And perhaps you were not misled by the faulty reasoning above. But the counterintuitive results and apparent paradoxes in probability routinely trip up even experts, and they will bewilder you soon enough. The only safe way to proceed is to lean lightly on intuition and heavily on the mathematical underpinnings of probability theory. In this lecture, we'll set up a framework for solving probability problems that will never lead you astray.

The second lesson is: don't play complicated betting games with a mathematician.

1 Analysis of Carnival Dice

Typically, probability theory is used to model some sort of randomized game, process, or experiment. If our model is good enough, we can mathematically derive additional properties of the game, process, or experiment that are not immediately apparent. For example, we'll shortly derive the fact that playing a lot of Carnival Dice is going to lose you a whole pile of money. We'll analyze the game using a four-step method that is a great way to approach a wide range of probability problems.

Step 1: Find the Sample Space

A probabilistic model has several components. The basic building block is called an *outcome*.

Definition 1 *An* outcome *of an experiment consists of the total information about the experiment after it has been performed, including the values of all random choices.*

Outcomes sometimes go by other names, such as *sample point*, *atomic event*, or *elementary event*. In Carnival Dice, there are three random choices: the number that comes up on the red die, the number on the green die, and the number on the blue die. Therefore, an outcome in Carnival Dice is an ordered triple of numbers, each in the range 1 to 6. For example, $(6, 3, 4)$ is the outcome where the red die comes up 6, the green die comes up 3, and the blue die comes up 4.

Definition 2 *The* sample space *for an experiment is the set of all possible outcomes.*

A *tree diagram* is a handy tool that can help you determine the sample space for an experiment. Here's how to build one. Begin with a single node. Create an outgoing branch to a new node for each value of the first random choice in the experiment. (In Carnival Dice, this is the number that comes up on the red die.) From each of these new nodes, create an outgoing branch to a another new node for each value of the second random choice. Continue in this way until there are no more random choices left in the experiment. Each leaf of the resulting tree corresponds to an outcome of the experiment. The tree diagram for Carnival Dice is shown in Figure 1.

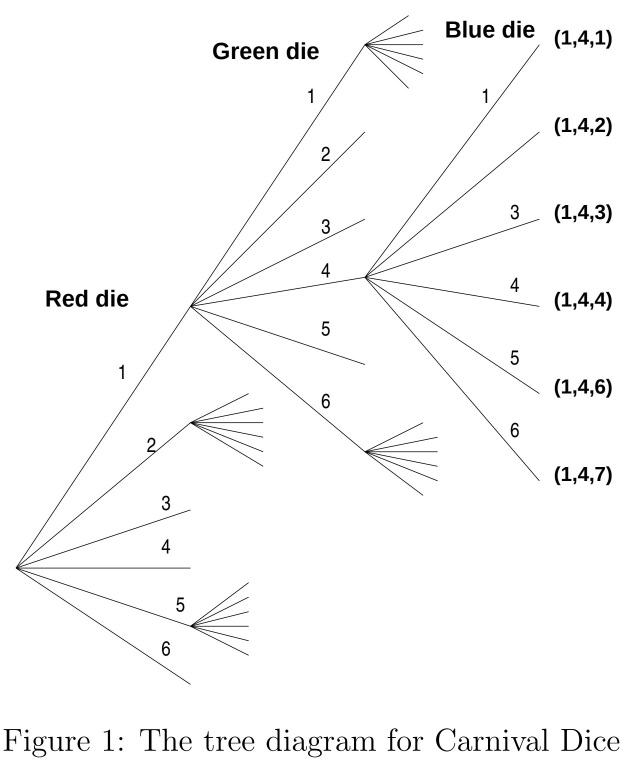

Figure 1: The tree diagram for Carnival Dice.

As is often the case, we can not draw the tree diagram for Carnival Dice completely, because there are too many outcomes. But we can get draw enough to see what the tree looks like and, consequently, to be able to reason about the problem effectively. In this case, the tree diagram suggests that the sample space S consists of all possible ordered triples of numbers, each in the range 1 to 6:

$$S = \{ (1,1,1),\ (1,1,2),\ (1,2,1),\ \ldots,\ (6,6,5),\ (6,6,6) \}$$

Using some mathematical notation[1], we can describe this sample space succinctly:

$$S = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$$

Step 2: Define Events of Interest

Usually we are interested not in the probability of a single outcome (such as rolling all ones), but in the probability of some more complicated event (such as you winning in Carnival Dice).

Definition 3 *An* event *is a subset of the sample space S.*

This is a neat definition. Everything that could possibly happen during an experiment can be characterized as a subset of the sample space and is, therefore, an event. For example,

$$\{ (1,1,2),\ (1,2,1),\ (2,1,1) \}$$

is the event that the sum of the numbers shown on the three dice is equal to four. You winning in Carnival Dice is also an event. We could describe it by explicitly listing all the outcomes that it contains, as we did above, but a more compact description is also possible:

$$W = \{(x,y,z) \mid x = 6 \vee y = 6 \vee z = 6\}$$

In words, the right side of this equation says, "the set of all triples (x,y,z) such that x is 6 or y is 6 or z is 6".

[1]The English language is full of words with ambiguous meanings, perhaps because it was evolved for hunting brontosaurus rather than for discussing intricate mathematical ideas. To remove these ambiguities, mathematicians communicate using a variety of odd symbols such as $\times$, $\vee$, $\forall$, and $\exists$. We shall use such symbols often in 6.042. A useful table that explains their meaning appears inside the front cover of the Rosen textbook and inside the head of your TA.

Step 3: Specify Outcome Probabilities

So far we've only tried to capture the set of things than can possibly happen in an experiment. The next job is to assign likelihoods to those things.

Definition 4 *A* probability space *consists of a sample space S and a* probability function $\Pr : S \mapsto \mathbb{R}$ *such that:*

1. $\forall s \in S \quad 0 \leq \Pr\{s\} \leq 1$
2. $\sum_{s \in S} \Pr\{s\} = 1$

The expression $\Pr\{s\}$ is read "the probability of s". Intuitively, $\Pr\{s\}$ is the fraction of the time that s is the outcome of the experiment, if the experiment is repeated many times. The two conditions on Pr listed above say, "every outcome gets a probability between 0 and 1" and "the sum of all outcome probabilities is 1".

Strictly speaking, Pr is an ordinary function, just like the ubiquitous f and g in calculus. But there are a couple caveats. In this class, we'll usually write $\Pr\{x\}$ instead of $\Pr(x)$, because often the thing in parentheses itself involves parentheses and that gets to be a mess. (That said, many authors use $\Pr(x)$ or $\Pr[x]$ or even $\Pr x$.) Furthermore, we'll shortly begin using the Pr notation in all sorts of contexts in which one would never use an ordinary function.

How do we assign a probability to each outcome in Carnival Dice? That is, for each outcome s in the sample space, what is the right value of $\Pr\{s\}$? Ultimately, there is no indisputably correct answer. We define $\Pr\{s\}$ in the way that best reflects our understanding of the problem. For example, if our experiment were flipping a quarter, we might assign probability $\frac{1}{2}$ to the outcome "heads" and probability $\frac{1}{2}$ to the outcome "tails". We hope that is a good assumption, but it is an *assumption*, not a empirically derived fact. After all, real quarters probably aren't perfectly balanced or aerodynamically uniform, and they might even land on edge sometimes. For most problems in this course— but not all— you will be told what assumptions to make about outcome probabilities.

Using a tree diagram, we can determine all outcome probabilities from relatively *few* assumptions about the experiment. Taking Carnival Dice as an example, let's assume that the red die shows each of its six faces with probability $\frac{1}{6}$. We represent this by writing $\frac{1}{6}$ on each branch out of the root node. Let's further assume that the green die also shows each of its faces with probability $\frac{1}{6}$, regardless of the outcome of the red die. So we write $\frac{1}{6}$ on each branch at the second level. Finally, let's assume that the blue die shows each face with probability $\frac{1}{6}$, regardless of the outcomes of the red and green dice. Again, we write $\frac{1}{6}$ on each branch at the third level. The resulting diagram is shown if Figure 2.

Now *the probability of an outcome is the product of the probabilities on the path from the root to that leaf.* For example, the path for outcome $(1, 4, 2)$ is highlighted in Figure 2. Using

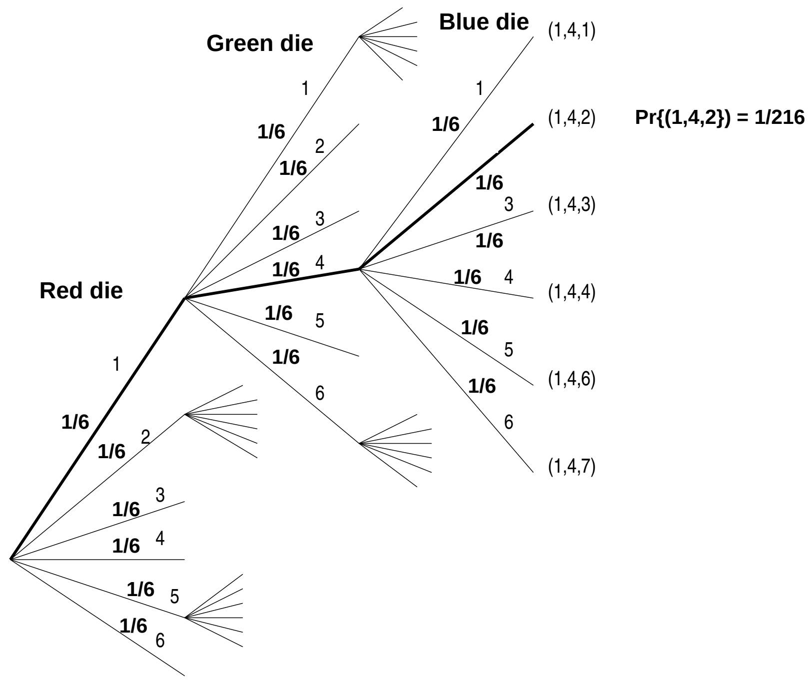

Figure 2: The tree diagram for Carnival Dice, annotated with probabilities.

this rule, we conclude that the probability of outcome $(1, 4, 2)$— and every other outcome in the sample space— is:

$$\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \quad = \quad \frac{1}{216}$$

When all outcomes in the sample space are equally likely, as is the case for Carnival Dice, the sample space is said to be *uniform*. Of course, many tree diagrams and sample spaces will not be so simple and symmetric!

Step 4: Compute Event Probabilities

We have now assigned each outcome a probability. What remains is to determine the probability of an *event*, such as the event that you win a round of Carnival Dice.

Definition 5 *The probability of an event A is written* $\Pr\{A\}$ *and is equal to*

$$\sum_{s \in A} \Pr\{s\}$$

In words, the probability of an event is the sum of the probabilities of the outcomes contained in that event. For example, the probability that the sum of the dice is four is:

$$\begin{aligned}
\Pr\{(1,1,2)\} + \Pr\{(1,2,1)\} + \Pr\{(2,1,1)\} &= \frac{1}{216} + \frac{1}{216} + \frac{1}{216} \\
&= \frac{1}{72}
\end{aligned}$$

Since each outcome in Carnival Dice has equal probability, determining the probability that you win boils down to counting the number of outcomes in the event:

$$W = \{(x,y,z) \mid x = 6 \vee y = 6 \vee z = 6\}$$

Counting these outcomes is a bit tricky. We'll look at a log of counting techniques later, but for now we'll use an ad hoc method. If the red die is a six, then you win however the remaining two dice come up. This gives $6 \cdot 6 = 26$ winning outcomes. If the red die is not a six, but the green die is, then the red die could come up 5 ways and the blue die 6 ways, giving $5 \cdot 6 = 30$ more winning outcomes. Finally, if the blue die is a six and the red and green dices are not, then the red and green dice can come up in $5 \cdot 5 = 25$ ways. Therefore, the total number of winning outcomes is $36 + 30 + 25 = 91$. Since each outcome has probability $\frac{1}{216}$, the probability of the event that you win at Carnival Dice is:

$$\begin{aligned}
\Pr\{W\} &= 91 \cdot \frac{1}{216} \\
&= \frac{91}{216} \\
&\approx 42\%
\end{aligned}$$

We can check this answer by computing the probability that you *lose*, which should be one minus the probability that you win. The number of losing outcomes in simply $5 \cdot 5 \cdot 5 = 125$, since each die can come up only 5 different ways without being a six. Therefore, the probability that you lose is:

$$\begin{aligned}
125 \cdot \frac{1}{216} &= \frac{125}{216} \\
&\approx 58\%
\end{aligned}$$

This is consistent with our previous answer. The probability that you win (91/216) plus the probability that you lose (125/216) is equal to 1, as it should be.

We can conclude that your odds of winning Carnival Dice are nowhere near even. In fact, they are worse than in any common casino game!