Quiz 2 Solutions

Problem 1. [16 pts] Next to each statement, circle either **true** or **false**.

1. **true** **false** $((A \to B) \land (B \to C) \land (C \to A)) \to (A \lor B \lor C)$ is a tautology.
2. **true** **false** $(\exists x\ \forall y\ P(x,y)) \to (\forall y\ \exists x\ P(x,y))$ is true for every domain of discourse and every predicate P.
3. **true** **false** The relation $\approx$ on the set $\mathbb{R} \times \mathbb{R}$ defined by $(x,y) \approx (x',y')$ iff $x = x'$ or $y = y'$ is an equivalence relation.
4. **true** **false** If $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ are both injective functions, then the function $h : \mathbb{N} \to \mathbb{N}$ defined by $h(n) = f(n) + g(n)$ is also injective.
5. **true** **false** If a finite directed graph contains no cycles, then some vertex has no outgoing edges. (A self-loop is considered a cycle.)
6. **true** **false** Let R_1 and R_2 be equivalence relations over a set S, and let $\sim$ be a relation over S such that $x \sim y$ iff $x\, R_1\, y$ and $x\, R_2\, y$. Then $\sim$ is also an equivalence relation.
7. **true** **false** Every acyclic, connected, undirected graph without self-loops and with exactly 175 vertices has exactly 174 edges.
8. **true** **false** Suppose that q and r are states of a state machine and $q \to r$ is a valid transition. If $P(q)$ is false and $P(r)$ is true, then P cannot be an invariant.

Solution.

1. **false.** If A, B, and C are all false, then the expression on the left side of the implication is true, but the expression on the right side is false. Therefore, the whole statement is false.
2. **true.** Suppose that the left side of the implication is true. Then there exists some x_0 such that $P(x_0, y)$ is true for all y. This implies that for all y, there exists an x (namely, x_0) such that $P(x,y)$ is true. Thus, the right side is holds. Therefore, the implication always holds.
3. **false.** The relation $\approx$ is not transitive. In particular, $(0,0) \approx (0,1)$ and $(0,1) \approx (1,1)$, but $(0,0) \not\approx (1,1)$.
4. **false.** Let $f(n) = n$ and define g as follows:

$$g(n) = \begin{cases} 100 - n & \text{(for } n \leq 100\text{)} \\ n & \text{(for } n > 100\text{)} \end{cases}$$

Then $h(0) = h(1) = \ldots = h(100) = 100$. Thus, f and g are injective, but h is not.

5. **false.** The empty graph (no nodes or vertices) contains no cycle, and all of its (zero) nodes have outgoing edges. It turns out that this was unintentionally a trick question as the statement is true for any graph other than the empty graph, so everybody got full credit for this question. For a nonempty graph, the statement is true: pick a starting vertex and start following directed edges (you can always follow directed edges because each node has an outgoing edge). If you ever revisit a node, the graph is cyclic; if you never do, it is not finite.
6. **true.** The relation $\sim$ is reflexive, symmetric, and transitive; one can check that it inherits all these properties from R_1 and R_2. If we take symmetry as an example, $x \sim y$ implies $y \sim x$, since $x\ R_1\ y$ and $x\ R_2\ y$ imply $y\ R_1\ x$ and $y\ R_1\ x$.
7. **true.** Every acyclic, connected, undirected graph is a tree. And for every tree, the number of vertices exceeds the number of edges by 1.
8. **false.** An invariant is something which, if it is true, will remain true as the state machine follows transitions. By this definition, it is perfectly possible for an invariant which is initially false to become true at a later time (it will then remain true forever).

Problem 2. **[14 pts]** Use induction to prove that the following inequality holds for all integers $n \geq 1$.

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \geq \frac{1}{2n}$$

Solution.

Proof. The proof uses induction. Let $P(n)$ be the proposition that:

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \geq \frac{1}{2n}$$

First, we prove $P(1)$. In this case, both side of the inequality are equal to $1/2$, and the inequality $1/2 \geq 1/2$ holds. Next, for each $n \geq 1$, we must show that $P(n)$ implies $P(n+1)$. Assume that $P(n)$ is true. Then we can reason as follows:

$$\begin{aligned} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n) \cdot (2n+2)} &\geq \frac{1}{2n} \cdot \frac{2n+1}{2n+2} \\ &> \frac{1}{2n+2} \end{aligned}$$

The first step uses the induction hypothesis, $P(n)$. The second step uses the fact that $(2n+1)/(2n) > 1$ for all $n \geq 1$. Therefore, by induction, the proposition $P(n)$ is true for all $n \geq 1$, and the claim is proved. ■

Problem 3. **[12 pts]** For the following problems, the domain of discourse is the set of all the people in the world. The predicate $L(x, y)$ holds if and only if x *truly* loves y.

(a) **[6 pts]** Translate the statement below into an English sentence that is as simple as you can make it.

$$\neg\exists x \; (\; \neg\exists y \; (L(y, x) \wedge x \neq y) \;)$$

Solution. Everyone is truly loved by someone else.

(b) **[6 pts]** Translate the statement below into logic notation.

There is exactly one person in the world that truly loves himself/herself.

In your translation, use only variables, the predicate L, and these symbols:

$$\exists \quad \forall \quad \neg \quad \wedge \quad \vee \quad \rightarrow \quad \leftrightarrow \quad (\quad , \quad) \quad =$$

Solution. Many solutions are possible. Here is one:

$$\exists x \; (L(x, x) \wedge \forall y \; (L(y, y) \rightarrow x = y))$$

Problem 4. **[14 pts]** In this problem, the term *graph* refers to an undirected graph without self-loops. Let $G = (V, E)$ be a graph such that every vertex has degree at most k. Suppose that $k + 1$ colors are available. Use induction to prove that there is a way to assign a color to each vertex of G so that for every edge in G, the two vertices joined by that edge are assigned different colors.

Solution. The proof is by induction on the number of vertices in G. Let $P(n)$ be the proposition that every n-vertex graph such that every vertex has degree at most k can be colored with $k + 1$ colors so that the endpoints of every edge are colored differently. First, note that $P(0)$ is true; if the graph contains no vertices, then the claim holds vacuously.

Next, for all $n \geq 0$, we must prove that $P(n)$ implies $P(n+1)$. Assume that $P(n)$ is true and consider a graph G with $n + 1$ vertices and maximum degree k. Pick one vertex $v \in V$. Let G' be the graph obtained from G by deleting the vertex v and all edges that touch v. Now, the degree of each vertex in G' is no greater than the degree of the corresponding vertex in

G. Therefore, the maximum degree in G' is at most k. Therefore, $P(n)$ implies that G' can be colored with $k+1$ colors. Color each vertex in G in the same way as the corresponding vertex in G'. All that remains is to color the vertex v. There are $k+1$ colors available, and v is joined by edges to at most k other vertices. Therefore, one can color v differently from all the vertices to which it is joined by an edge. Therefore, $P(n+1)$ holds, and the claim is proved by induction.

Problem 5. [**12 pts**] Solve the following problems using the Pigeonhole Principle.

(a) [**6 pts**] A 100-point exam will be given to a class with 128 students. The exam can be designed so that every student receives a score in the range from k to 100. How large must k be to ensure that at least three students receive the same score? Specify the "pigeons" and the "holes" used in your argument.

Solution. The pigeons are the 128 exams, and the holes are the possible scores from k to 100. Each exam is assigned to the score it receives. There are $100-k+1$ possible score. By the Pigeonhole Principle, three students must receive the same score if:

$$\begin{aligned} 128 &\geq 2 \cdot (100 - k + 1) + 1 \\ k &\geq \frac{75}{2} \end{aligned}$$

Since k must be an integer, we must choose $k \geq 38$.

(b) [**6 pts**] A dresser drawer in a dark room contains socks of n different colors. In particular, there are $k_i \geq 2$ socks of the i-th color for $1 \leq i \leq n$. What is the minimum number of socks that must you take from the drawer to be certain that you get at least one matching pair? Again, specify the "pigeons" and the "holes".

Solution. The pigeons are the socks taken from the drawer, and the holes are the n colors. Each sock is assigned to its color. By the Pigeonhole Principle, if $n+1$ socks are taken, then there must be two socks of the same color. Taking n socks would not be sufficient as each sock could then be of a different color. Therefore, $n+1$ is the minimum number of socks that must be taken from the drawer.

Problem 6. [**16 pts**] Each monk entering the Temple of Forever is given a bowl with 15 red beads and 12 green beads. Each time the Gong of Time rings, a monk must do one of two things:

1. If he has at least 3 red beads in his bowl, then he may remove 3 red beads and add 2 green beads.
2. He may replace each green bead in his bowl with a red bead and replace each red bead in his bowl with a green bead.

A monk may leave the Temple of Forever only when he has exactly 5 red beads and 5 green beads in his bowl.

(a) [**4 pts**] Model the life of a monk in the Temple of Forever as a state machine. Specify a set of states Q, a set of start states Q_0, and a transition relation δ.

Solution. The set Q of states is $\mathbb{N}\times\mathbb{N}$. The set of start states is $Q_0 = \{(15, 12)\}$. The transition relation δ contains the following transitions for every state $(r, g) \in Q$.

$$\begin{array}{rcll}(r,g) & \rightarrow & (r-3, g+2) & \qquad\qquad (\text{if } r \geq 3) \\ (r,g) & \rightarrow & (g,r) & \end{array}$$

(b) [**4 pts**] Let r be the number of red beads in a monk's bowl, and let g be the number of green beads. Is the property "$r \neq g$" an invariant? Justify your answer.

Solution. No. The property holds when the monk has 8 red beads and 3 green beads in his bowl. After a transition of the first type, he has 5 red beads and 5 green beads so the property no longer holds. Therefore, the this property is not an invariant.

(c) [**4 pts**] Is the property "$r - g$ is equal to $5k - 3$ or $5k + 3$ for some $k \in \mathbb{Z}$" an invariant? Justify your answer.

Solution. Yes. Suppose that the property holds for state (r, g). Then there exists a k such that $r - g$ is equal to $5k + 3$ or $5k - 3$. Now we must check that the property is maintained under the two types of transition.

1. After a transition of the first type, the property still holds. If $r - g = 5k + 3$, then $g - r = 5(-k) - 3$; if $r - g = 5k - 3$, then $g - r = 5(-k) + 3$.
2. After a transition of the second type, the property still holds. If $r - g = 5k + 3$, then $(r - 3) - (g + 2) = 5(k - 1) + 3$; if $r - g = 5k - 3$, then $(r - 3) - (g + 2) = 5(k - 1) - 3$.

Therefore, the property is an invariant.

(d) [**4 pts**] Use the Invariant Theorem to prove no one ever leaves the Temple of Forever.

Solution. The invariant above holds for the start state, since $15 - 12 = 5 \cdot 0 + 3$. However, it does not hold for a bowl with 5 red beads and 5 green beads, because $5 - 5 = 0$ is not of the form $5k \pm 3$. Therefore, this state is not reachable by the Invariant Theorem.

Problem 7. [**16 pts**] This problem concerns solutions to the following equation:

$$4x^3 + 2y^3 = z^3$$

(a) [**3 pts**] Suppose that x, y, and z are positive integers that satisfy the equation. Show that z must be even.

Solution. If x and y are integers, then $4x^3 + 2y^3$ is even. Therefore, z^3 is even. If z was odd, then z^3 would be odd, thus we must conclude that z is even.

(b) [**3 pts**] Show that y must be even.

Solution. Since z is even, z^3 is a multiple of 4. This means that $z^3 - 4x^3$ is a multiple of 4. Therefore, $2y^3$ is a multiple of 4. This means that y^3 is even, and so y is even (y can't be odd or y^3 would be odd as well.

(c) [**3 pts**] Show that x must be even.

Solution. Since z and y are even, $z^3 - 2y^3$ is a multiple of 8. Therefore, $4x^3$ is a multiple of 8. This means that x^3 is even. Therefore, x is even (x can't be odd or x^3 would be odd as well).

(d) [**7 pts**] Use the well-ordering principle to prove that there are no positive integer solutions to the equation:

$$4x^3 + 2y^3 = z^3$$

Solution. Suppose that the claim is false. Let S be the set of positive integers x for which there exist positive integers y and z that solve the equation. By assumption, S is not empty. Therefore, by the well-ordering principle, S contains a smallest element x'. Let y' and z' be corresponding values of y and z that give a solution. By the preceding arguments, x', y', and z' must be even. Thus $x'/2$, $y'/2$, and $z'/2$ are all positive integers. Dividing both sides of the original equation by 2^3 shows that these values also satisfy the equation. Therefore $x'/2$ is in S, contraditing the definition of x' as the smallest element of S. Therefore, the orginal supposition is wrong, and the claim is true.