Quiz 1 Solutions

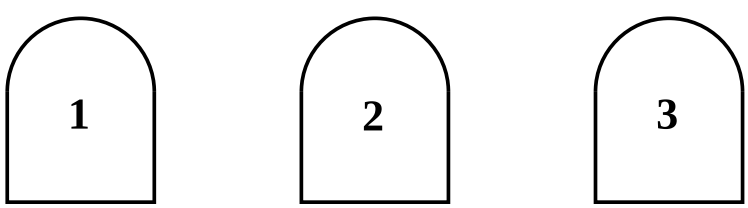
Problem 1. **[20 pts]** Next to each statement, circle either **true** or **false**.

1. **true** **false** If A and B are events and $\Pr\{A \cap B\} = 0$, then they are necessarily independent.
2. **true** **false** $1 + 3 + 3^2 + 3^3 + \ldots + 3^{99} \geq 3^{100}/2$.
3. **true** **false** For every pair of random variables X and Y, $\text{Ex}[X - Y] = \text{Ex}[X] - \text{Ex}[Y]$.
4. **true** **false** If X and Y are independent random variables, then $\text{Ex}[X \cdot Y] = \text{Ex}[X] \cdot \text{Ex}[Y]$.
5. **true** **false** In five independent flips of a fair coin, the sequence of results $TTHTH$ is more likely than the sequence $HHHHH$.
6. **true** **false** For all independent random variables X and Y such that $\Pr\{Y \neq 0\} = 1$, $\text{Ex}[X/Y] = \text{Ex}[X]/\text{Ex}[Y]$.
7. **true** **false** For all events A and B, $\Pr\{A \cup B\} + \Pr\{A \cap B\} = \Pr\{A\} + \Pr\{B\}$.
8. **true** **false** For all events A and B such that $A \subseteq B$ and $\Pr\{B\} > 0$, $\Pr\{A\} = \Pr\{A \mid B\} \cdot \Pr\{B\}$.
9. **true** **false** For every random variable A with expectation x, we have $\Pr\{A = x\} > 0$.
10. **true** **false** For all random variables A, B, and C (not necessarily independent), if $\text{Ex}[A] > \text{Ex}[B]$ and $\text{Ex}[B] > \text{Ex}[C]$, then $\text{Ex}[A] > \text{Ex}[C]$.

Solution. The correct answers are as follows:

1.	false	6.	false
2.	false	7.	true
3.	true	8.	true
4.	true	9.	false
5.	false	10.	true

Problem 2. **[15 pts]** A Barglesnort makes its lair in one of three caves:



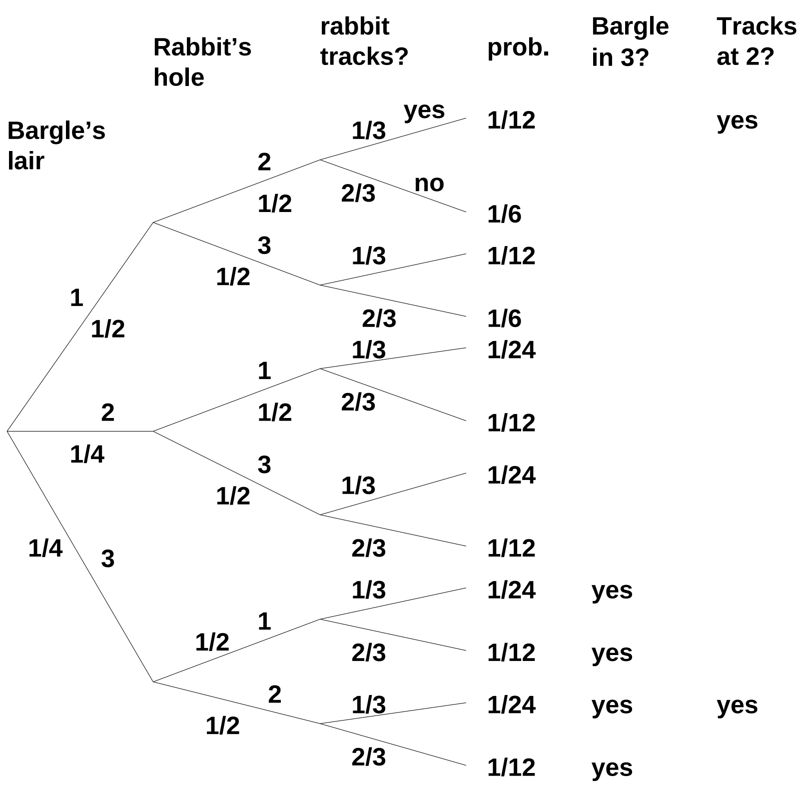
The Barglesnort inhabits cave 1 with probability $\frac{1}{2}$, cave 2 with probability $\frac{1}{4}$, and cave 3 with probability $\frac{1}{4}$. A rabbit subsequently moves into one of the two unoccupied caves, selected uniformly at random. With probability $\frac{1}{3}$, the rabbit leaves tracks at the entrance to its cave. (Barglesnorts are much too clever to leave tracks.) What is the probability that the Barglesnort lives in cave 3, given that there are no tracks in front of cave 2?

If you fail to draw a tree diagram, then no partial credit will be awarded for an incorrect solution.

Solution. A tree diagram is given below. Let B_3 be the event that the Barglesnort inhabits cave 3, and let T_2 be the event that there are tracks in front of cave 2. Taking data from the tree diagram, we can compute the desired probability as follows:

$$\begin{aligned}\Pr\left\{B_3 \mid \overline{T_2}\right\} &= \frac{\Pr\left\{B_3 \cap \overline{T_2}\right\}}{\Pr\left\{\overline{T_2}\right\}} \\ &= \frac{\frac{1}{24}+\frac{1}{12}+\frac{1}{12}}{1-\frac{1}{12}-\frac{1}{24}} \\ &= \frac{5}{21}\end{aligned}$$

In the denominator, we apply the formula $\Pr\left\{\overline{T_2}\right\} = 1 - \Pr\{T_2\}$ for convenience.



Problem 3. [**15 pts**] Answer the following questions about independence.

(a) [**5 pts**] Suppose that you roll a fair die that has six sides, numbered 1, 2, ..., 6. Is the event that the number on top is a multiple of 2 independent of the event that the number on top is a multiple of 3? Justify your answer.

Solution. Let A be the event that the number on top is a multiple of 2, and let B be the event that the number on top is a multiple of 3. We have:

$$\Pr\{A\} \cdot \Pr\{B\} = \frac{3}{6} \cdot \frac{2}{6} = \frac{1}{6} = \Pr\{A \cap B\}$$

Therefore, these events are independent.

(b) [**5 pts**] Now suppose that you roll a fair die that has *four* sides, numbered 1, 2, 3, 4. Again, is the event that the number on top is a multiple of 2 independent of the event that the number on top is a multiple of 3? Justify your answer.

Solution. As before, let A be the event that the number on top is a multiple of 2, and let B be the event that the number on top is a multiple of 3. Now, however, we have:

$$\begin{aligned} \Pr\{A\} \cdot \Pr\{B\} &= \frac{2}{4} \cdot \frac{1}{4} \\ &= \frac{1}{8} \end{aligned}$$

But:

$$\Pr\{A \cap B\} = 0$$

Since these results disagree, the events are not independent.

(c) [**5 pts**] Finally, suppose that you roll a fair die that has *eight* sides, numbered 1, 2, ..., 8. Let the random variable X be the remainder when the number on top is divided by 2, and let the random variable Y be the remainder when the number on top is divided by 3. Are the random variables X and Y independent? Justify your answer.

Solution. First, let's tabulate the values of X and Y:

die roll	X	Y
1	1	1
2	0	2
3	1	0
4	0	1
5	1	2
6	0	0
7	1	1
8	0	2

Working from the table, we have:

$$\Pr\{X = 1 \cap Y = 1\} = \frac{2}{8}$$

But:

$$\begin{aligned} \Pr\{X = 1\} \cap \Pr\{Y = 1\} &= \frac{4}{8} \cdot \frac{3}{8} \\ &= \frac{3}{16} \end{aligned}$$

Since these results conflict, the random variables are not independent.

Problem 4. **[15 pts]** Each 6.042 quiz 1 is graded according to a rigorous procedure:

- With probability $\frac{4}{7}$ the exam is graded by a *tutor*, with probability $\frac{2}{7}$ it is graded by a *lecturer*, and with probability $\frac{1}{7}$, it is accidentally dropped behind the radiator and arbitrarily given a score of 84.
- *Tutors* score an exam by scoring each problem individually and then taking the sum.
  - There are ten true/false questions worth 2 points each. For each, full credit is given with probability $\frac{3}{4}$, and no credit is given with probability $\frac{1}{4}$.
  - There are four questions worth 15 points each. For each, the score is determined by rolling two fair dice, summing the results, and adding 3.
  - The single 20 point question is awarded either 12 or 18 points with equal probability.
- *Lecturers* score an exam by rolling a fair die twice, multiplying the results, and then adding a "general impression" score.
  - With probability $\frac{4}{10}$, the general impression score is 40.
  - With probability $\frac{3}{10}$, the general impression score is 50.

– With probability $\frac{3}{10}$, the general impression score is 60.

Assume all random choices during the grading process are mutually independent.

(a) [**5 pts**] What is the expected score on an exam graded by a tutor?

Solution. Let the random variable T denote the score a tutor would give. By linearity of expectation, the expected sum of the problem scores is the sum of the expected problem scores. Therefore, we have:

$$\begin{aligned}
\mathrm{Ex}[T] &= 10 \cdot \mathrm{Ex}[\text{T/F score}] + 4 \cdot \mathrm{Ex}[\text{15pt prob score}] + \mathrm{Ex}[\text{20pt prob score}] \\
&= 10 \cdot \left(\frac{3}{4} \cdot 2 + \frac{1}{4} \cdot 0\right) + 4 \cdot \left(2 \cdot \frac{7}{2} + 3\right) + \left(\frac{1}{2} \cdot 12 + \frac{1}{2} \cdot 18\right) \\
&= 10 \cdot \frac{3}{2} + 4 \cdot 10 + 15 \\
&= 70
\end{aligned}$$

(b) [**5 pts**] What is the expected score on an exam graded by a lecturer?

Solution. Now we find the expected value of L, the score a lecturer would give. Employing linearity again, we have:

$$\begin{aligned}
\mathrm{Ex}[L] &= \mathrm{Ex}[\text{product of dice}] + \mathrm{Ex}[\text{general impression}] \\
&= \left(\frac{7}{2}\right)^2 + \left(\frac{4}{10} \cdot 40 + \frac{3}{10} \cdot 50 + \frac{3}{10} \cdot 60\right) \\
&= \frac{49}{4} + 49 \\
&= 61\frac{1}{4}
\end{aligned}$$

(c) [**5 pts**] What is the expected score on a 6.042 exam?

Solution. Let X equal the true exam score. The total expectation law implies:

$$\begin{aligned}
\mathrm{Ex}[X] &= \frac{4}{7} \cdot \mathrm{Ex}[T] + \frac{2}{7} \cdot \mathrm{Ex}[L] + \frac{1}{7} \cdot 84 \\
&= \frac{4}{7} \cdot 70 + \frac{2}{7} \cdot \left(\frac{49}{4} + 49\right) + \frac{1}{7} \cdot 84 \\
&= 40 + \frac{7}{2} + 14 + 12 \\
&= 69\frac{1}{2}
\end{aligned}$$

Problem 5. [**20 pts**] A spacecraft will be lost if and only if a critical component fails. The craft contains 400 critical components that are "highly reliable". Each of these components fails with probability 1 in 1000. The remaining 5,500 critical components on board are "extremely reliable". Each of these fails with probability 1 in 10,000. Let p be the probability that the spacecraft is lost.

(a) [**8 pts**] If component failures occur mutually independently, what is the probability p that the spacecraft is lost? Your answer may be an expression, instead of a number.

Solution. Let H_i be the event the that i-th highly-reliable component fails, and let X_j be the event that the j-th extremely-reliable component fails. Then we have:

$$\begin{aligned}
p &= \Pr\left\{H_1 \cup H_2 \cup \ldots \cup H_{400} \cup X_1 \cup X_2 \cup \ldots \cup X_{5500}\right\} \\
&= 1 - \Pr\left\{\overline{H_1 \cup H_2 \cup \ldots \cup H_{400} \cup X_1 \cup X_2 \cup \ldots \cup X_{5500}}\right\} \\
&= 1 - \Pr\left\{\overline{H_1} \cap \overline{H_2} \cap \ldots \cap \overline{H_{400}} \cap \overline{X_1} \cap \overline{X_2} \cap \ldots \cap \overline{X_{5500}}\right\} \\
&= 1 - \Pr\left\{\overline{H_1}\right\} \cdot \Pr\left\{\overline{H_2}\right\} \cdots \Pr\left\{\overline{H_{400}}\right\} \cdot \Pr\left\{\overline{X_1}\right\} \cdot \Pr\left\{\overline{X_2}\right\} \cdots \Pr\left\{\overline{X_{5500}}\right\} \\
&= 1 - \prod_{i=1}^{400} \Pr\left\{\overline{H_i}\right\} \cdot \prod_{j=1}^{5500} \Pr\left\{\overline{X_j}\right\} \\
&= 1 - \prod_{i=1}^{400} (1 - \Pr\{H_i\}) \cdot \prod_{j=1}^{5500} (1 - \Pr\{X_j\}) \\
&= 1 - \left(1 - \frac{1}{1000}\right)^{400} \cdot \left(1 - \frac{1}{10000}\right)^{5500}
\end{aligned}$$

The first step uses the defintion of p. In the second step, we apply the identity $\Pr\{E\} = 1 - \Pr\{overlineE\}$. Then we apply DeMorgan's law. The fourth step uses the fact that component failures are mutually independent. In the fifth step, we simply switch to product notation. Then, once again, we apply the rule $\Pr\{E\} = 1 - \Pr\{overlineE\}$. In the last step, we substitute in the component failure probabilities.

(b) [**6 pts**] Now do not assume that components fail mutually independently. Give the tightest upper bound you can on p. Justify your answer.

Solution. Intuitively, if events are cover to as much of the sample space as possible, then they ought not to overlap one another. In other worse, the events should be disjoint. Formally, if the failures are disjoint events, then the sum rule says:

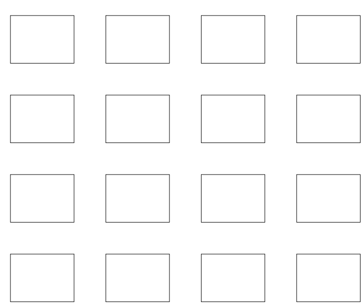
$$\begin{aligned} p &= \Pr\{H_1 \cup H_2 \cup \ldots \cup H_{400} \cup X_1 \cup X_2 \cup \ldots \cup X_{5500}\} \\ &= \Pr\{H_1\} + \Pr\{H_2\} + \ldots + \Pr\{H_{400}\} + \\ &\quad \Pr\{X_1\} + \Pr\{X_2\} + \ldots + \Pr\{X_{5500}\} \\ &= 400 \cdot \frac{1}{1000} + 5500 \cdot \frac{1}{10000} \\ &= \frac{95}{100} \end{aligned}$$

This is actally tight. Boole's inequality says that the probability of a union of events is at most the sum of the event probabilities.

(c) [**6 pts**] Again, do not assume that components fail mutually independently. Give the tightest lower bound you can on p. Justify your answer.

Solution. Certainly, the probability of failure is at least 1/1000, since a single each highly reliable component fails with that probability. On the other hand, the probability could actually be no higher than that, since all the other events could be a subset of that one.

Problem 6. [**15 pts**] A classroom has sixteen desks arranged as shown below.



If there is a girl in front, behind, to the left, or to the right of a boy, then the two of them *flirt*. One student may be in multiple flirting couples; for example, a student in a corner of the classroom can flirt with up to two others, while a student in the center can flirt with as many as four others.

Suppose that desks are occupied by boys and girls with equal probability and mutually independently. What is the expected number of flirting couples? Justify your answer.

Solution. First, let's count the number of pairs of adjacent desks. There are three in each row and three in each column. Since there are four rows and four columns, there are $3 \cdot 4 + 3 \cdot 4 = 24$ pairs of adjacent desks.

Number these pairs of adjacent desks from 1 to 24. Let F_i be an indicator for the event that occupants of the desks in the i-th pair are flirting. The probability we want is then:

$$\begin{aligned}
\text{Ex}\left[\sum_{i=1}^{24} F_i\right] &= \sum_{i=1}^{24} \text{Ex}\left[F_i\right] \\
&= \sum_{i=1}^{24} \Pr\left\{F_i = 1\right\}
\end{aligned}$$

The first step uses linearity of expectation, and the second step uses the fact that the expecatation of an indicator is equal to the probability that it is 1.

The occupants of adjacent desks are flirting if the first holds a girl and the second a boy or vice versa. Each of these events happens with probability $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, and so the probability that the occupants flirt is $\Pr\left\{F_i = 1\right\} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. Plugging this into the previous expression gives:

$$\begin{aligned}
\text{Ex}\left[\sum_{i=1}^{24} F_i\right] &= \sum_{i=1}^{24} \Pr\left\{F_i = 1\right\} \\
&= 24 \cdot \frac{1}{2} \\
&= 12
\end{aligned}$$