Final Exam

Write your name here and on each page: ______________________________

Circle the name of your tutor:

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- This exam is closed book. Calculators and computers are not allowed.
- You may use two handwritten pages of notes. Photocopies are not acceptable.
- Show your work as fully and as clearly as possible so that partial credit may be given.
- Write your solutions in the space provided. If you run out of space, use the back of the page.
- Do not discuss this exam with students who have not yet taken it. Do not leave with a copy of the exam under any circumstances.

Good luck!

Problem	Points	Score
1	16	
2	12	
3	20	
4	10	
5	10	
6	12	
7	10	
8	10	
TOTAL	100	

Problem 1. **[16 pts]** Next to each statement, circle either **true** or **false**.

1. **true** **false** If an invariant holds for a state q in a state machine, then q must be reachable from the start state.

2. **true** **false** For every $n \in \mathbb{N}$, there exists an $m \in \mathbb{N}$ such that:

$$\sum_{k=1}^{m} \frac{1}{k^{1.01}} \quad > \quad \sum_{k=1}^{n} \frac{1}{k}$$

3. **true** **false** Every undirected graph without self-loops contains at least two vertices with the same degree.

4. **true** **false** Let X be a real-valued random variable defined on a finite sample space. Then $\Pr\{X \geq \operatorname{Ex}[X]\} > 0$.

5. **true** **false** There exists an injective function $f : A \to B$ where A and B are finite sets and $|A| > |B|$.

6. **true** **false** If I is an indicator random variable, then $\operatorname{Ex}[I^2] = \Pr\{I = 1\}$.

7. **true** **false** Suppose that $X \to Y$ is true. Then the number of propositions below that are true must be either one or three.

$$Y \to X \qquad\qquad \neg X \to \neg Y \qquad\qquad \neg Y \to \neg X$$

8. **true** **false** Let X, Y, and Z be mutually independent events that occur with probability $1/2$. Then $\Pr\left\{(X \cup Y \cup Z) \cap \overline{(X \cap Y \cap Z)}\right\} = 1/2$.

Problem 2. **[12 pts]** In a tournament digraph, a vertex v is called an *emperor* if there are directed edges from v to every other vertex in the tournament. A *cycle* in a tournament is a nonempty sequence of directed edges $v_1 \to v_2$, $v_2 \to v_3$, $v_3 \to v_4$, $\ldots$, $v_k \to v_1$. A tournament is *acyclic* if it has no cycles. Use induction or well-ordering to prove that every nonempty, acyclic tournament has an emperor.

Problem 3. **[20 pts]** This problem consists of five short-answer questions. Your answers may contain unevaluated factorials and binomial coefficients. No explanations are required, but partial credit may be awarded for work shown.

(a) A five-card hand is dealt from a well-shuffled 52-card deck. What is the probability that the hand contains exactly three aces?

(b) In a binary string, a *transition* is a pair of consecutive bits that are different from one another. How many n-bit strings have exactly k transitions?

(c) A parachute maker produces 270 parachutes, of which, in retrospect, 13 were defective. Assume that the parachutes were made one at a time. What is the largest integer n such that the parachute maker must have produced at least n non-defective parachutes in a row?

(d) A cube is *colorful* if every face is painted either red, orange, yellow, green, blue, or indigo and no two faces are painted the same color. Two colorful cubes are considered the same if one can be obtained from the other by rotation. How many different colorful cubes are possible?

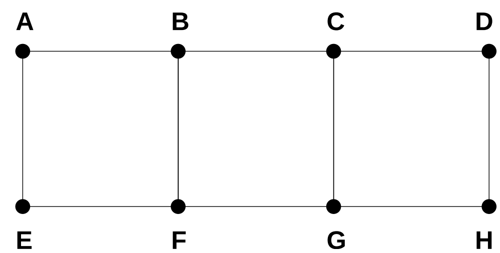
(e) Six couples are seated around a circular table. The two people in each couple must sit beside one another. Two seatings are considered the same if one can be obtained from the other by rotation. How many different seating arrangements are possible?

Problem 4. **[10 pts]** Four children are playing ball: Amy, Bill, Carol, and Stinky Peterson. Amy has the ball initially. Whenever a child has the ball, he or she throws it to one of the other three children selected uniformly and independently at random.

(a) What is the expected number of throws before Stinky Peterson touches the ball for the first time? Justify your answer.

(b) What is the expected number of throws before everyone has touched the ball at least once? Justify your answer.

Problem 5. **[10 pts]** Suppose that each edge in the graph below is colored either red or blue uniformly and independently at random. A *simple cycle* in a graph is a sequence of distinct edges $(v_1, v_2), (v_2, v_3), (v_3, v_4), \ldots, (v_k, v_1)$. What is the expected number of simple cycles that contain only blue edges? Justify your answer.



Problem 6. **[12 pts]** Let S be a permutation of the symbols 0, 1, 2, ..., 9 selected uniformly at random. For example, we might have:

$$S = 2198476530$$

Justify your answers to the following questions.

(a) What is the probability that S contains the substring 60?

(b) What is the probability that S contains the substrings 42 and 60?

(c) What is the probability that S contains the substrings 03 and 60?

(d) What is the probability that S contains at least one of the following substrings?

$$60 \qquad 42 \qquad 03$$

Problem 7. **[10 pts]** Every day, Dangerous Dan wakes up and begins his daily ordeal:

- He stumbles down the front steps, taking a fatal fall with probability 1/5.
- If he survives the steps, then he wanders across the street, getting hit by a bus with probability 1/8.
- If he manages to avoid the bus, then he blunders into work, where he dies of acute carpal tunnel syndrome with probability 1/3.

He repeats this ordeal every day until he dies. Dangerous Dan just woke up. What is the probability that the bus runs him over before he fatally falls or types himself to death?

Problem 8. **[10 pts]** Give a combinatorial proof of the following identity.

$$\binom{4n+1}{2n+1} = \sum_{k=-n}^{n} \binom{2n-k}{n}\binom{2n+k}{n} \qquad \text{(for } n \in \mathbb{N}\text{)}$$