Proofs: Valid or Invalid?

Determine whether the following proofs are valid. If a proof is wrong, pinpoint the precise error. If it is correct, justify each step.

Problem 1.

Claim: The number $\lg 3 = \log_2 3$ is irrational.

Proof. A number is irrational if it is not the ratio of two integers. Therefore, since $\lg 3$ cannot be written in the form a/b, where a and b are integers, it is irrational. ∎

Problem 2.

Claim: For any real number n, if $n^2 \leq 9$ then, $n \leq 3$.

Proof. If $n > 3$, then $n^2 > 9$. ∎

Problem 3.

Claim: If a and b are two equal real numbers, then $2 = 1$.

Proof.

$$\begin{aligned} a &= b \\ a^2 &= ab \\ a^2 - b^2 &= ab - b^2 \\ (a-b)(a+b) &= (a-b)b \\ a+b &= b \\ 2b &= b \\ 2 &= 1 \end{aligned}$$

∎

Problem 4.

Claim: There exist irrational numbers a and b such that a^b is a rational number. (You may use the fact that $\sqrt{2}$ is irrational).

Proof. Note that $\sqrt{2}$ is irrational. Now, consider the value $\sqrt{2}^{\sqrt{2}}$. If this quantity is rational, then the claim holds for $a = \sqrt{2}$ and $b = \sqrt{2}$. On the other hand, if the quantity $\sqrt{2}^{\sqrt{2}}$ is not rational, then the claim holds for $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. ∎