

A Appendix

A.1 Combinatorics

A.1.1 Counting

Given a set of n elements:

- number of r -permutations without repetition:

$$P(n, r) ::= n(n-1) \dots (n-r+1) = \frac{n!}{(n-r)!}$$

- r -combinations without repetition:

$$C(n, r) ::= \binom{n}{r} = \frac{n!}{(n-r)! r!}$$

- r -permutations with unlimited repetition (r distinct balls, n distinct bins):

$$n^r$$

- r -combinations with repetition (r identical balls, n distinct bins):

$$\binom{n+r-1}{r}$$

- permutations with limited repetition:

$$\binom{n}{r_1, r_2, \dots, r_k} ::= \frac{n!}{r_1! r_2! \dots r_k!}$$

where $r_1 + r_2 + \dots + r_k = n$.

A.1.2 Binomial Identity

$$\binom{n}{m} = \binom{n}{n-m}$$

A.1.3 Multinomial Theorem

$$(x_1 + x_2 + \dots + x_k)^n = \sum_{r_1+r_2+\dots+r_k=n} \binom{n}{r_1, r_2, \dots, r_k} x_1^{r_1} x_2^{r_2} \dots x_k^{r_k}$$

A.1.4 Inclusion-Exclusion

$$|A_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k+1} \left(\sum_{S \subseteq \{1, \dots, n\}, |S|=k} \left| \bigcap_{i \in S} A_i \right| \right)$$

A.2 Events

$$\Pr \left\{ \bigcup_{n \in \mathbb{N}} A_n \right\} = \sum_{n \in \mathbb{N}} \Pr \{A_n\} \quad \text{for pairwise disjoint } A_n \quad (\text{Sum Rule})$$

$$\Pr \{A - B\} = \Pr \{A\} - \Pr \{A \cap B\} \quad (\text{Difference Rule})$$

$$\Pr \{\overline{B}\} = 1 - \Pr \{B\} \quad (\text{Complement Rule})$$

$$\Pr \{A \cup B\} = \Pr \{A\} + \Pr \{B\} - \Pr \{A \cap B\} \quad (\text{Inclusion-Exclusion})$$

$$\Pr \{A \cup B\} \leq \Pr \{A\} + \Pr \{B\} \quad (\text{Boole's inequality})$$

$$\Pr \{A\} \leq \Pr \{B\}, \quad \text{for } A \subseteq B \quad (\text{Monotonicity})$$

$$\Pr \{A \cap B\} \leq \Pr \{A\},$$

$$\Pr \{A\} \leq \Pr \{A \cup B\}$$

$$\Pr \{A \mid B\} \Pr \{B\} = \Pr \{A \cap B\} \quad (\text{Product Rule})$$

$$\Pr \{A \mid B\} = \frac{\Pr \{B \mid A\} \Pr \{A\}}{\Pr \{B\}} \quad (\text{Bayes Rule})$$

Lemma (Law of Total Probability). Let B_0, B_1, \dots be disjoint events whose union is the entire sample space. Then for all events A ,

$$\Pr \{A\} = \sum_{i \in \mathbb{N}} \Pr \{A \cap B_i\},$$

$$\Pr \{A\} = \sum_{i \in \mathbb{N}} \Pr \{A|B_i\} \Pr \{B_i\}. \quad (\text{conditional form})$$

A.3 Independence

Definition. Events A and B are *independent* iff

$$\Pr \{A \cap B\} = \Pr \{A\} \Pr \{B\}.$$

Events A_0, A_1, A_2, \dots are *mutually independent* iff for all subsets $J \subseteq \mathbb{N}$,

$$\Pr \left\{ \bigcap_{i \in J} A_i \right\} = \prod_{i \in J} \Pr \{A_i\}.$$

Events A_0, A_1, A_2, \dots are *k-wise independent* iff $\{A_i \mid i \in J\}$ are mutually independent for all subsets $J \subset \mathbb{N}$ with $|J| = k$.