Mini-Quiz 1

1. Write your name:

2. (Rosen 1.2.13). Determine whether $(\neg q \wedge (p \implies q)) \implies \neg p)$ is a tautology.

3. Determine the truth value of the following propositions, where the universe of discourse is the natural numbers.

   (a) $\forall n \exists m (n^2 < m)$

   (b) $\exists m \forall n (n^2 < m)$

   (c) $\exists n \exists m (n^2 + m^2 = 5)$

   (d) $\forall n, m \exists p (p = (m + n)/2)$

Tutorial 1 Problems

Problem 1. Identify *precisely* what the bugs are in each of the following false proofs.

(a)

$$\begin{aligned} 3 &> 2 \\ 3\log_{10}(1/2) &> 2\log_{10}(1/2) \\ \log_{10}(1/2)^3 &> \log_{10}(1/2)^2 \\ (1/2)^3 &> (1/2)^2 \\ \text{Therefore, } 1/8 &> 1/4! \end{aligned}$$

(b) You are richer than you think:

$$1¢ = \$0.01 = (\$0.1)^2 = (10¢)^2 = \$1.$$

(c) Consider the following scheme to make millions. Obtain a slab of gold $10'' \times 11'' \times 1''$. Then cut out two little one-inch triangles at the corners. These triangles are the profit. Finally, slide the two slabs together again and note that the rectangle is now $11'' \times 10'' \times 1''$—the same size as it was before we removed the small triangles. By repeating the process, we can produce an unlimited supply of gold!

(d) Claim: Every polynomial $p(x) = ax^2 + bx + c$ over $\mathbb{C}$ (the complex numbers) has two roots.

Sounds good, but the claim is false. Find the counterexamples and then restate the claim so it is correct.

Proof. Using the usual formulas for the roots of a quadratic, let

$$\begin{aligned} r_1 &::= \frac{-b+\sqrt{D}}{2a}, \text{ and} \\ r_2 &::= \frac{-b-\sqrt{D}}{2a}, \text{ where} \\ D &::= b^2 - 4ac. \end{aligned}$$

To verify that r_1 is a root, note that

$$\begin{aligned} a{r_1}^2 &= a(\frac{-b+\sqrt{d}}{2a})^2 \\ &= a\frac{b^2 - 2b\sqrt{D} + D}{4a^2} \\ &= \frac{b^2 - 2b\sqrt{D} + b^2 - 4ac}{4a} \\ &= \frac{2b^2 - 2b\sqrt{D} - 4ac}{4a} \\ &= \frac{b^2 - b\sqrt{D} - 2ac}{2a} \end{aligned}$$

and

$$\begin{aligned} br_1 &= b\frac{-b+\sqrt{D}}{2a} \\ &= \frac{-b^2 + b\sqrt{D}}{2a} \end{aligned}$$

so

$$\begin{aligned} p(r_1) &= {r_1}^2 + br_1 + c \\ &= \frac{b^2 - b\sqrt{d} - 2ac}{2a} + \frac{-b^2 + b\sqrt{d}}{2a} + c \\ &= \frac{-2ac}{2a} + c = -c + c = 0, \end{aligned}$$

as claimed. A similar calculation confirms that $p(r_2) = 0$. ∎

Problem 2. A *group* is a set G together with a binary operation $\cdot : G \times G \to G$ that satisfies the following axioms.

Identify at least two "normal" sets and operations on them that satisfy the group axioms.

1. $\forall a, b, c \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c)$.
2. $\exists e \in G$ (called the *identity*) such that $\forall a \in G, a \cdot e = e \cdot a = a$.
3. $\forall a \in G, \exists a^{-1} \in G$ (called the *inverse* of a), such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

So addition defines a group on the integers, and multiplication a group on the positive rationals.

Using the three group axioms prove the following propositions.

(a) Show, by contradiction, that the identity element is unique.

(b) Show that for every $a \in G$, the inverse of a is unique. We can therefore write the inverse element as a^{-1} with no fear of ambiguity.

(c) Show $\forall a \in G, (a^{-1})^{-1} = a$

(d) Show $\forall a, b \in G, (a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$

Problem 3. Use induction on n to prove that the following inequality holds for all integers $n \geq 1$ and for all non-negative real numbers $x_1, \ldots, x_n$.

$$(x_1 + x_2 + \ldots + x_n)^2 \quad \geq \quad x_1^2 + x_2^2 + \ldots + x_n^2$$

Problem 4. *Handshakes.* I recently went to a party with my spouse; there were 11 other married couples there. A lot of handshaking happened (but of course no one shook hands with their spouse). At the end, it turned out that for every k from 0 to 22, someone other than me had shaken hands with exactly k people. How many people did I shake hands with? Prove your answer.

Note that there are 24 people (12 couples) in this case.