Quiz 1

Your Name: ______________________________

Circle the name of your Tutorial Instructor:

- This quiz is open book. There are XXX problems totaling XXX points. Problem parts are labelled with their point values.
- Put your name on the top of **every** page — *these pages may be separated for grading.*
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- You may assume any of the results from class or notes in any proofs you give.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. **Be neat and write legibly**.
- GOOD LUCK!

Write the hour at which your Friday recitation begins ______

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	20		
2	15		
3	20		
4	20		
5	25		
Total	100		

Problem 1 [pts] **Logic**

It is time to take pictures at Richard Nixon Elementary School. In Mrs. Tjaded's class are six students: Carl, David, Josh, Kyle, Rob, and Shishir. Naturally, the students are arguing, so in order to minimize the disruption on the waiting line, Mrs. Tjaded has to order the students according to the following rules:

a) Carl must come before Josh.

b) David must be directly in front of or directly behind Kyle

c) Shishir must be the fifth person on line.

d) Josh cannot be last.

(a) Which of the following people could be last on line? Explain your reasoning.

(1) Shishir

(2) Kyle

(3) Rob

(4) Carl

(5) David

(b) Suppose we add the constraint, "Carl must be second." Prove that there is no way to line up the kids.

Problem 2 [pts] **Relations**

(a) Let $F(x,t)$ denote "you can fool person x at time t." Write in logical notation "you can fool all the people some of the time, and some of the people all the time, but you can't fool all the people all the time." Assume "some of" means "at least one."

(b) What is the common name for the reflexive closure of the "less than" relation?

(c) What is the common name for the transitive closure of the "child of" relation?

(d) What is the common name for the inverse of the "child of" relation?

(e) Consider the partial order "aRb if a divides b without remainder." What are the minimal elements of this relation called?

(f) Let $C(x,y)$ denote "x is a child of y." What is the common name for the relation $C^{-1} \circ C^{-1} \circ C \circ C$?

Problem 3 [pts] **State Machines**

Consider the following algorithm on graphs:

> Consider a graph G in which each vertex is either uncolored, or is colored red or blue. Initially all vertices except one—call this vertex s—are uncolored, while s is red.
> At each step, one of two events can occur:
>
> a) Choose an uncolored vertex v which is adjacent to a red vertex, and color v blue.
>
> b) Choose an uncolored vertex v which is adjacent to a blue vertex, and color v red.
>
> If neither rule can be applied, the algorithm halts. If all vertices are colored, it halts in a "well done" state; otherwise it halts in a "stuck" state.

(a) Formalize this algorithm as a state machine.

(b) Prove that the algorithm eventually halts.

(c) Prove the following fact: on any path connecting a colored vertex and an uncolored vertex, there must be some edge incident on both a colored vertex and an uncolored vertex.

(d) Using the fact you proved in the previous part, prove that if G is connected, all vertices are colored when the algorithm halts.

(e) Consider applying this algorithm to *bipartite* graphs, i.e., graphs in which the vertices can be partitioned into two sets R and B such that there is no edge between any pair of vertices in R (and similarly for B). Suppose the initial state has some vertex in r colored red. Using the invariant theorem, prove that in any state, only vertices of R are colored red and only vertices of B are colored blue.

(f) Combine the previous parts to prove that when run on a connected bipartite graph, the above algorithm halts and outputs a valid 2-coloring.

Problem 4 [pts] **Relations**

Burton read the induction proof for the 2-coloring problem from the second problem set, and "discovered" another interesting property. The original problem was

> Suppose you take a piece of paper and draw a bunch of straight lines, each in its own arbitrary direction, that completely cross the paper. This divides the paper up into polygonal regions. Prove by induction that you can always color the various regions using only *two* colors, so that any two regions that share a boundary line are different colors. Note that this need not be true of regions that share only a boundary point.

Burton defined a relation between regions as follows:

$$xRy ::= x = y, \text{ or region } x \text{ does not border region } y$$

Burton claims R is an equivalence relation. Is it? If so, prove it; if not, provide a counterexample and explain why it is a counterexample.

Problem 5 [pts] **Induction**

Use induction to prove that

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots + (n-2)(n-1)n = \frac{1}{4}(n+1)n(n-1)(n-2)$$

whenever n is a natural number.

Problem 6 [pts] **Induction**

The denizens of the land of Frohike have a passion for masochism: their currency—the Eebae™—uses coins only of values 5, 7, and 11 Eebaes.

The Frohikies have discovered over time that any value of at least 14 Eebaes can be produced using some combination of these coins, but some values below 14 are not possible.

Prove by induction that all values of at least 14 Eebaes can be made from these three coin values.

Problem 7 [pts] **Structural Induction**

Consider the sets $S_1, S_2, \ldots$ defined inductively as

- $S_0 = \{0, 1\}$
- For all $n \geq 1$, $S_n = \left\{ \frac{x+y}{2} \mid x, y \in S_{n-1} \right\}$

i.e., where S_n contains the average values of each pair of numbers in S_{n-1}.

(a) Write out the elements of each of the following sets: S_1, S_2, and S_3. Generalize (in your own words or in symbols) what the elements of S_n are.

(b) Prove by induction that for all n, $1/3 \notin S_n$. You may not assume your generalization from the previous part is true; you must prove that statement if you wish to use it.