Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
Professors David Karger and Nancy Lynch

Quiz 1
March 8 2000

Quiz 1

Your Name: ______________________

Circle the name of your Tutorial Instructor:

Carl	David	Josh	Ken
Kyle	Robert	Shishir	Victor

Write the hour at which your Friday recitation begins ______

- This quiz is open book/notes.
- There are 11 pages with 7 problems totaling 70 points. Problem parts are labeled with their point values.
- Put your name on the top of **every** page — *these pages may be separated for grading.*
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- You may assume any of the results presented in class or in the lecture notes.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. **Be neat and write legibly**.
- GOOD LUCK!

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	10		
2	10		
3	10		
4	10		
5	10		
6	5		
7	15		
Total	70		

Problem 1 [10 points] **Quantifiers**

Consider the following predicates where the universe of discourse is the set of all people:

$P(x)$**:** x is a hacker

$Q(x)$**:** x is an MIT student

$R(x)$**:** x is an artist

$S(x)$**:** x is willing to waltz

Express each of the following statements using quantifiers, logical connectives, and and the above predicates:

(a) [1 point] No hackers are willing to waltz.

(b) [1 point] No artists are unwilling to waltz.

(c) [1 point] All MIT students are hackers.

(d) [1 point] MIT students are not artists.

(e) [4 points] Does (d) follow logically from (a), (b), and (c)? If so, give a proof. If not, give a counterexample.

(f) [1 point] Translate into English: $(\forall x)(R(x) \vee S(x) \implies Q(x))$.

(g) [1 point] Translate into English: $(\exists x)(R(x) \wedge \neg Q(x)) \implies (\forall x)(P(x) \implies S(x))$.

Problem 2 [10 points] **Induction**

Prove **by induction** (other methods will not receive credit) that

$$(1 \cdot 2) + (2 \cdot 3) + (3 \cdot 4) + \cdots + (n-1)n = \frac{1}{3}(n-1) \cdot n \cdot (n+1)$$

whenever n is a natural number greater than 1.

Problem 3 [10 points] **Structural Induction**

Consider the sets $S_1, S_2, \ldots$ defined inductively as

- $S_0 = \{0, 1\}$
- For all $n \geq 1$, $S_n = \left\{\frac{x+y}{2} \mid x, y \in S_{n-1}\right\}$

i.e., where S_n contains the average values of each pair of numbers in S_{n-1}. Note we allow $x = y$ in the definition of S_n.

(a) [1 point] Write out the elements of each of the following sets: S_1, S_2, and S_3.

(b) [2 points] Generalize (in your own words or in symbols) to give the elements of S_n.

(c) [7 points] Prove by induction that for all $n \in \mathbb{N}$, $1/3 \notin S_n$. You will probably want to strengthen the induction hypothesis. You may not assume your generalization from the previous part is true; you must prove that statement if you wish to use it.

Problem 4 [10 points] **Relations**

(a) [1 point] What is the common name for the reflexive closure of the "less than" relation?

(b) [1 point] What is the common name for the transitive closure of the "child of" relation?

(c) [2 points] What is the common name for the inverse of the "teaches" relation (that is, if xTy means x teaches y, how do we say $aT^{-1}b$)?

(d) [3 points] Consider the partial order "aRb iff a divides b without remainder" on the universe of natural numbers **greater than 1**. What are the minimal elements under this relation usually called?

(e) [3 points] Is the relation "xRy iff $x = y$ or x is **not** y's roommate" an equivalence relation? Why or why not?

Problem 5 [10 points] **Precedence Relations**

Death is planning the end of the world. This involves a number of tasks each of which takes one minute to complete. The prerequisites associated with these tasks are listed below.

Abbrv.	Task	Prerequisites
C	Compose a requiem	
N	Notify the UN	B
D	Signal the daemons	B
B	Blow the trumpet	
T	Sell T-shirts: "All I got was this lousy t-shirt."	N
Q	Grade the 6.042 quiz	S
G	Open the gates	D,N
S	Put out the sun	C,N

(a) [5 points] Represent the tasks and their prerequisites as a directed graph.

(b) [5 points] Death is omnipotent and can therefore work on as many tasks at a time as he wishes. What is the minimum amount of time required for him to end the world? Why? (An explanation without proof is fine.)

Problem 6 [5 points] **Graphs**

Let $G = (V, E)$ be a simple, undirected graph, and let C be a set of colors. Define a *partial coloring of G using C* to be a function that assigns to each node in V either a color in C or no color. (That is, it colors some, none, or all of the nodes using colors in C.) Define a *partly colored graph* to be a simple, undirected graph $G = (V, E)$ together with a partial coloring of G.

Prove the following fact about partly colored graphs:

> In a partly colored graph, any walk connecting a colored node to an uncolored node must include a colored node adjacent to an uncolored node.

Problem 7 [15 points] **Algorithms**

Let $G = (V, E)$ be a simple, undirected graph. The following algorithm, *RedBlue*, manipulates partial colorings of G using $C = \{\text{red}, \text{blue}\}$. (Partial colorings are defined in the previous problem.)

> Initially, one vertex r of G is colored red, and all the other vertices are uncolored. At each step, one of two events occurs:
>
> a) Some uncolored vertex v that is adjacent to a red vertex becomes blue.
>
> b) Some uncolored vertex v that is adjacent to a blue vertex becomes red.
>
> If neither rule can be applied, the algorithm terminates.

(a) [3 points] Formalize this algorithm as a state machine; that is, define Q, Q_0, and δ.

(b) [3 points] Prove that the algorithm eventually terminates.

(c) [3 points] Using the fact stated in Problem 6, prove that if G is connected, then all vertices are colored when the algorithm terminates.

(d) [3 points] Consider applying this algorithm to *bipartite* graphs, i.e., graphs in which the vertices can be partitioned into two sets A and B such that there is no edge between any pair of vertices in A, and similarly for B. Suppose the node r that is colored red in the initial state is in set A. Using the Invariant Theorem, prove that in any reachable state, only vertices of A are colored red and only vertices of B are colored blue.

(e) [3 points] Combine the previous parts to prove that when run on a connected bipartite graph, the above algorithm terminates and outputs a valid 2-coloring.