Prove: $\forall n \geq 0(P(n))$

1. (Base) $P(0)$ — There's exactly one empty string of 0s and 1s.
2. (Inductive step) $\forall n \geq 0(P(n) \Rightarrow P(n+1))$
   1. Fix $n \geq 0$.
   2. Assume $P(n)$, that is, there are 2^n length n strings.
   3. $P(n+1)$, that is, there are 2^{n+1} length $n+1$ strings. — Every string of length n can have either a 0 or a 1 attached at the end.
   4. QED — Implication, UG
3. QED — Induction

Prove: $\forall n \geq 0(P(n))$

1. (Base) $P(0)$
2. (Inductive step) $\forall n \geq 0[(\forall i, 0 \leq i \leq n(P(n))) \Rightarrow P(n+1)]$
3. QED Strong induction

Prove: $\forall n \geq k(P(n))$

1. (Base) $P(k)$
2. (Inductive step) $\forall n \geq k[(\forall i, k \leq i \leq n(P(n))) \Rightarrow P(n+1)]$
3. QED Strong induction

Prove: $\forall n \geq 12(P(n))$

1. (Base) $P(12)$ Use 3+3+3+3
2. (Inductive step) $\forall n \geq 12[(\forall i, 12 \leq i \leq n(P(n))) \Rightarrow P(n+1)]$
3. QED Strong induction

2. (Inductive step) $\forall n \geq 12[(\forall i, 12 \leq i \leq n(P(n))) \Rightarrow P(n+1)]$
    1. Fix $n \geq 12$.
    2. $(\forall i, 12 \leq i \leq n(P(n))) \Rightarrow P(n+1)$
        1. Assume $\forall i, 12 \leq i \leq n(P(n)$
        2. $P(n+1))$
            1. $n = 12 \Rightarrow P(n+1)$ Use 7+3+3
            2. $n = 13 \Rightarrow P(n+1)$ Use 7+7
            3. $n \geq 14 \Rightarrow P(n+1)$???
            4. QED Cases
        3. QED Implication
    3. QED UG

3. $n \geq 14 \Rightarrow P(n+1)$

1. Assume $n \geq 14$.	
2. $P(n-2)$	Ind. hyp. 2.2.1 with $n-2 \geq 12$
3. $P(n+1)$	Add one block to the tower from the previous step.
4. QED	Implication

Prove: $\forall n \geq 12(P(n))$

1. (Base) $P(12)$
2. (Base) $P(13)$
3. (Base) $P(14)$
4. (Inductive step) $\forall n \geq 14[(\forall i, 12 \leq i \leq n(P(n))) \Rightarrow P(n+1)]$
3. QED

Very specialized strong induction

Prove: $\forall n \geq 0(P(n))$

1. (Base) $P(0)$ — $F_0 = 0$, which is even.
2. (Inductive step) $\forall n \geq 0[(\forall i, 0 \leq i \leq n(P(n))) \Rightarrow P(n+1)]$ — ???
3. QED — Strong induction

Prove: $\forall n \geq 2(P(n))$

1. (Base) $P(2)$ $\quad F_2 = 1 = (\frac{3}{2})^0$
2. (Base) $P(3)$ $\quad F_3 = 2 \geq (\frac{3}{2})^1$
3. (Inductive step) $\forall n \geq 3[(\forall i, 2 \leq i \leq n(P(i))) \Rightarrow P(n+1)]$
    1. Fix $n \geq 3$
    2. Assume $\forall i, 2 \leq i \leq n(P(i))$
    3. $P(n+1)$ $\quad$???
    4. QED $\quad$ Implication, UG
4. QED $\quad$ Strong induction

3. $P(n+1)$
    1. $F_n \geq (\frac{3}{2})^{n-2}$ — Inductive hyp. 3.2
    2. $F_{n-1} \geq (\frac{3}{2})^{n-3}$ — Inductive hyp. 3.2
    3. $F_{n+1} = F_n + F_{n-1}$ — Definition
    4. $F_{n+1} \geq (\frac{3}{2})^{n-2} + (\frac{3}{2})^{n-3}$ — Plug inequalities into equation
    5. QED — Algebra
        1. $\frac{3}{2} + 1 \geq (\frac{3}{2})^2$ — Arithmetic
        2. $(\frac{3}{2})^{n-2} + (\frac{3}{2})^{n-3} \geq (\frac{3}{2})^{n-1}$ — Algebra
        3. $F_{n+1} \geq (\frac{3}{2})^{n-1}$ — 3.3.4 and 3.3.5.2
        4. QED — Def. of $P(n+1)$

Prove: $\forall n \geq 2(P(n))$

1. $P(2)$ Use 2 itself
2. $\forall n \geq 2[(\forall i, 2 \leq i \leq n)(P(i)) \Rightarrow P(n)]$
   1. Fix $n \geq 2$
   2. Assume $\forall i, 2 \leq i \leq n(P(i))$, that is, all numbers up to and including n can be factored into primes
   3. $P(n+1)$, that is, $n+1$ can be factored into primes. ???
   4. QED Implication, UG
3. QED Strong induction

3. $P(n+1)$, that is, $n+1$ can be factored into primes.

   1. If $n+1$ is prime, then $n+1$ can be factored into primes.
      Use $n+1$ itself

   2. If $n+1$ is not prime, then $n+1$ can be factored into primes.

      1. Assume $n+1$ is not prime.
      2. $n+1$ can be written as $a \times b$, where $1 < a, b < n+1$
         Def. of "prime"
      3. a can be factored into primes. Strong ind. hyp. (2.2)
      4. b can be factored into primes. Strong ind. hyp. (2.2)
      5. $n+1$ can be factored into primes.
         Use factors of a and b.
      6. QED Implication

   3. QED Cases

Prove: $\forall n \geq 3(P(n))$

1. (Base) $P(3)$ — Trivial by assumption.
2. (Inductive step) $\forall n \geq 3(P(n) \Rightarrow P(n+1))$
   1. Fix $n \geq 3$
   2. Assume $P(n)$
   3. $P(n+1)$
      1. Fix T, a tournament, and $p_1 \to p_2 \to \ldots \to p_{n+1} \to p_1$, an $n+1$-cycle in T.
      2. T contains a 3-cycle. — ???
      3. QED — Implication, UG
   4. QED — Implication, UG
3. QED

2. T contains a 3-cycle.

   1. If $p_3 \to p_1$ then T contains a 3-cycle.

      Use $p_1 \to p_2 \to p_3 \to p_1$.

   2. If $p_1 \to p_3$ then T contains a 3-cycle.

      1. Assume $p_1 \to p_3$.
      2. T contains an n-cycle.

         Use $p_1 \to p_3 \to p_4 \to p_{n+1} \to p_1$.

      3. T contains a 3 cycle.

         Inductive hypothesis (2.2)

   3. QED — Cases

Prove: $\forall n \geq 1(P(n))$

1. (Base) $P(1)$ Obvious
2. (Inductive step) $\forall n \geq 1[(\forall i, 1 \leq i \leq n(P(n))) \Rightarrow P(n+1)]$
    1. Fix $n \geq 1$
    2. Assume $\forall i, 1 \leq i \leq n(P(i))$.
    3. $P(n+1)$
        1. Fix any $n+1$-player tournament T.
        2. T has a locally fair ranking.
            1. Choose x to be any player,
               B the set of players that beat x, and
               L the set of players that lose to x.
            2. B has a locally fair ranking. Ind. hyp. (2.2).
            3. L has a locally fair ranking. Ind. hyp. (2.2).
            4. Define a ranking R of all the players by:
               B ranking, then x, then L ranking.
            5. R is a locally fair ranking for T.
            6. QED EG
        3. QED UG
    4. QED Implication, UG
3. QED Strong induction

Prove: $\forall n \geq 1(P(n))$

1. (Base) $P(1)$ — Only one possibility.
2. (Inductive step) $\forall n \geq 1[P(n) \Rightarrow P(n+1)]$
    1. Fix $n \geq 1$.
    2. Assume $P(n)$.
    3. $P(n+1)$.
        1. Fix T, an $n+1$-player tournament,
           and z, a player with the most wins.
        2. $\exists$ a locally fair ranking of T with z first. — ???
        3. QED — UG
    4. QED — Implication, UG
3. QED — Induction

2. $\exists$ a locally fair ranking of T with z first.
    1. If z is undefeated in T then $\exists$ a locally fair ranking of T with z first.
        1. Assume z is undefeated in T.
        2. Choose R to be a locally fair ranking of all-but-z. Prev. thm and EI
        3. z then R is a locally fair ranking of T.
        4. $\exists$ a locally fair ranking of T with z first. EG
        5. QED Implication
    2. If z is defeated by someone then $\exists$ a locally fair ranking of T with z first. ???
    3. QED Cases

2. If z is defeated by someone then $\exists$ a locally fair ranking of T with z first.

   1. Assume z is defeated by someone.
   2. Choose x to be someone who defeated z.
   3. z has the best record in the subtournament involving all-but-x.
      z has best record in T, no wins are removed.
   4. $\exists$ ranking of subtournament having z first.
      Inductive hypothesis, z has best record.
   5. Choose R to be a ranking of all-but-x having z first.
      EI
   6. In T, x is defeated by someone.
      z was defeated, and x's record in T is not better than z's.
   7. Define ranking S to be R, with x inserted right after the lowest-ranked player that beats x.
   8. S is a locally fair ranking of T.
      R is locally fair, x is beaten by the player ranked just above it and beats all players ranked below it.
   9. z is first in S. z is first in R; x is not inserted at the top.
   10. $\exists$ a locally fair ranking of T with z first.
      S works, EG

   QED Implication