Prove: $\sqrt{2}$ is irrational.

1. $\sqrt{2}$ rational $\Rightarrow$ *false*
    1. Assume $\sqrt{2}$ rational.
    2. *false*
        1. Choose $a, b \in N^+$, $\sqrt{2} = \frac{a}{b}$, lowest terms. — Basic properties of $\mathbf{Q}$, EI
        2. $2b^2 = a^2$ — Algebra
        3. a^2 is even. — EG, def. of "even"
        4. a is even. — ???
        5. Choose $c \in N$, $a = 2c$. — Def. of "even", EI
        6. $4c^2 = a^2 = 2b^2$ — Algebra
        7. $2c^2 = b^2$ — Algebra
        8. b^2 is even. — EG, def. of "even"
        9. b is even. — ???
        10. a and b not in lowest terms. — 1.2.4, 1.2.9
        11. QED — 1.2.1, 1.2.10, logic
    3. QED — Implication.
2. QED — Contradiction.

The proof is by contradiction. Assume for purpose of contradiction that $\sqrt{2}$ is rational. Then we can write $\sqrt{2} = a/b$ where a and b are integers and the fraction is in lowest terms. Squaring both sides gives $2 = a^2/b^2$ and so $2b^2 = a^2$. This implies that a is even; that is, a is a multiple of 2. As a result, a^2 is a multiple of 4. Because of the equality $2b^2 = a^2$, $2b^2$ must also be a multiple of 4. This implies that b^2 is even and so b must be even. But since a and b are both even, the fraction a/b is not in lowest terms. This is a contradiction, and so the assumption that $\sqrt{2}$ is rational must be false.

To prove: $\forall n \geq 0(P(n))$

It's enough to show:

1. (Base) $P(0)$
2. (Inductive step) $\forall n \geq 0(P(n) \Rightarrow P(n+1))$

To prove: $\forall n \geq k(P(n))$

It's enough to show:

1. (Base) $P(k)$
2. (Inductive step) $\forall n \geq k(P(n) \Rightarrow P(n+1))$

Prove: $\forall n \geq 0(1+2+3+\ldots+n = \frac{n(n+1)}{2})$

Prove: $\forall n \geq 0(P(n))$

1. (Base) $P(0)$
2. (Inductive step) $\forall n \geq 0(P(n) \Rightarrow P(n+1))$
3. QED Induction

Prove: $\forall n \geq 0(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$.

1. (Base) $\sum_{i=1}^{0} i = \frac{0(0+1)}{2}$.
2. (Inductive step) $\forall n \geq 0(\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \Rightarrow \sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2})$
3. QED Induction

1. (Base) $\sum_{i=1}^{0} i = \frac{0(0+1)}{2}$ $(P(0))$
   1. $\sum_{i=1}^{0} i = 0$ — Def. of sum with no terms
   2. $\frac{0(0+1)}{2} = 0$ — Arithmetic
   3. QED — By 1.1 and 1.2.

2. (Inductive step) $\forall n \geq 0 (\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \Rightarrow \sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2})$

   1. Fix $n \geq 0$.
   2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \Rightarrow \sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$
      1. Assume $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
      2. $\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$???
      3. QED Implication
   3. QED UG

2. $\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$

   1. $\sum_{i=1}^{n+1} i = (\sum_{i=1}^{n} i) + (n+1)$ Sum has one extra term.
   2. $(\sum_{i=1}^{n} i) + (n+1) = \frac{n(n+1)}{2} + n + 1$ Inductive hypothesis $P(n)$ (2.2.1)
   3. $\frac{n(n+1)}{2} + n + 1 = \frac{(n+1)(n+2)}{2}$ Algebra
   4. QED Combine previous 3 equations.

Prove: $\forall n \geq 0\ (P(n))$

1. (Base) $P(0)$
2. (Inductive step) $\forall n \geq 0\ (P(n) \Rightarrow P(n+1))$
    1. Fix $n \geq 0$
    2. $P(n) \Rightarrow P(n+1)$
        1. Assume $P(n)$
        2. $P(n+1)$ — ???
        3. QED — Implication
    3. QED — Universal generalization
3. QED — Induction

Prove: $\forall n \in N, n \geq 0$ $(6|n^3 - n)$

1. (Base) $6|0^3 - 0$
2. (Inductive step) $\forall n \geq 0(6|n^3 - n \Rightarrow 6|(n+1)^3 - (n+1))$
   1. Fix $n \geq 0$.
   2. Assume $6|n^3 - n$
   3. $6|(n+1)^3 - (n+1)$???
   4. QED Implication, UG (condensing two steps here)
3. QED Induction

Prove: $\forall n \in N, n \geq 0(6|n^3 - n)$

1. (Base) $6|0^3 - 0$
2. (Inductive step) $\forall n \geq 0(6|n^3 - n \Rightarrow 6|(n+1)^3 - (n+1))$
    1. Fix $n \geq 0$.
    2. Assume $6|n^3 - n$
    3. $6|(n+1)^3 - (n+1)$
        1. $(n+1)^3 - (n+1) = n^3 - n + 3(n^2 + n)$ — Algebra
        2. $n^2 + n$ is even — Last lecture
        3. $2|n^2 + n$ — Definition of |
        4. $6|3(n^2 + n)$ — Multiplying both sides by 3
        5. $6|(n^3 - n) + 3(n^2 + n)$ — Divides both terms (2.2, 2.3.4), so divides sum.
        6. QED — 2.3.1, 2.3.5
    4. QED
3. QED

Prove: $\forall n \geq 0(\Sigma_{i=0}^{n} F_i^2 = F_n F_{n+1})$

1. (Base) $\Sigma_{i=0}^{0} F_i^2 = F_0 F_1$
2. (Inductive step) $\forall n \geq 0(\Sigma_{i=0}^{n} F_i^2 = F_n F_{n+1}$
   $\Rightarrow \Sigma_{i=0}^{n+1} F_i^2 = F_{n+1} F_{n+2})$???
3. QED Induction

Prove: $\forall n \geq 0 (\Sigma_{i=0}^{n} F_i^2 = F_n F_{n+1})$

1. (Base) $\Sigma_{i=0}^{0} F_i^2 = F_0 F_1$ — Both sides are 0.
2. (Inductive step) $\forall n \geq 0 (\Sigma_{i=0}^{n} F_i^2 = F_n F_{n+1} \Rightarrow \Sigma_{i=0}^{n+1} F_i^2 = F_{n+1} F_{n+2})$
    1. Fix $n \geq 0$.
    2. Assume $\Sigma_{i=0}^{n} F_i^2 = F_n F_{n+1}$
    3. $\Sigma_{i=0}^{n+1} F_i^2 = F_{n+1} F_{n+2}$
        1. $\Sigma_{i=0}^{n+1} F_i^2 = \Sigma_{i=0}^{n} F_i^2 + F_{n+1}^2$ — Sum has one extra term
        2. $\Sigma_{i=0}^{n} F_i^2 + F_{n+1}^2 = F_n F_{n+1} + F_{n+1}^2$ — Inductive hyp.
        3. $F_n F_{n+1} + F_{n+1}^2 = F_{n+1}(F_n + F_{n+1})$ — Algebra
        4. $F_{n+1}(F_n + F_{n+1}) = F_{n+1} F_{n+2}$ — Fibonacci definition
        5. QED — Combining equations
    4. QED — Implication, UG
3. QED — Induction

Prove: $\forall n \geq 3$ (Sum of interior angles of any n-sided convex polygon is $(n-2)180$ degrees)

1. (Base) Sum of angles in any triangle is 180. — Basic geometry
2. (Inductive step) $\forall n \geq 3\ (P(n) \Rightarrow P(n+1))$
    1. Fix $n \geq 3$.
    2. Assume sum of angles of any n-sided convex poly is $(n-2)180$.
    3. Sum of angles of any $n+1$-sided convex poly is $(n-1)180$.
        1. Fix any $n+1$-vertex convex poly X, say with vertices $x_1, x_2, \ldots, x_{n+1}$
        2. Let Y be poly with vertices $x_1, x_2, \ldots, x_n$ (cut out a vertex).
        3. Y is a convex poly with at least 3 vertices.
        4. Sum of angles of Y is $(n-2)180$. — Inductive hyp.
        5. Sum of angles of triangle $T = x_n, x_{n+1}, x_1$ is 180.
        6. Sum of angles in $X =$ sum in $Y +$ sum in T $= (n-2)180 + 180 = (n-1)180$. — Basic geometry, arithmetic.
        7. QED — UG
    4. QED — Implication, UG
3. QED — Induction

Prove: $\forall n \geq 0\ (P(n))$

1. (Base) $P(0)$
   1. $\forall k(f(0,k) = \frac{(0+k)!}{0!k!})$ — Both sides are 1
   2. QED — Definition of $P(0)$
2. (Inductive step) $\forall n \geq 0(P(n) \Rightarrow P(n+1))$
   1. Fix $n \geq 0$.
   2. Assume $P(n)$, that is, row n is correct.
   3. $P(n+1)$, that is, row $n+1$ is correct. — ???
   4. QED — Implication, UG
3. QED — Induction

3. $P(n+1)$
    1. $\forall k(Q(n+1,k))$
        1. (Base) $Q(n+1,0)$
        2. (Inductive step) $\forall k \geq 0(Q(n+1,k) \Rightarrow Q(n+1,k+1))$.
        3. QED — Induction
    2. QED — Definition of $P(n+1)$

1. (Base) $Q(n+1, 0)$
    1. $f(n+1, 0) = \frac{(n+1+0)!}{n+1!0!}$ — Both sides are 1
    2. QED — Definition of $Q(n+1)$

2. (Inductive step) $\forall k \geq 0(Q(n+1,k) \Rightarrow Q(n+1,k+1))$.
   1. Fix $k \geq 0$.
   2. Assume $Q(n+1,k)$.
   3. $Q(n+1,k+1)$
      1. $f(n+1,k+1) = f(n,k+1) + f(n+1,k)$
         Definition of $f(n+1,k+1)$
      2. $f(n+1,k) = \frac{(n+1+k)!}{(n+1)!k!}$ Inductive hypothesis 2.3.1.2.2, def. of $Q(n+1,k)$.
      3. $f(n,k+1) = \frac{(n+k+1)!}{n!(k+1)!}$ Inductive hypothesis 2.2, def. of $P(n,k+1)$.
      4. $f(n+1,k+1) = \frac{(n+k+1)!}{n!(k+1)!} + \frac{(n+1+k)!}{(n+1)!k!} = \frac{(n+1+k+1)!}{(n+1)!k+1!}$
         Combine equations, algebra
      5. QED Definition of $Q(n+1,k+1)$
   4. QED Induction