Lecture Notes

In this lecture we will cover methods for evaluating sums.
Sums arise very frequently in the analysis of algorithms and systems.
The main examples in this lecture, however, are drawn from finance and physics.

We've already introduced summation notation.
It is often a convenient way to write down an equation, but not so convenient when you actually want to evaluate it.
We are going to develop some tools for simplifying such sums.
One attack is to find a *closed form* for the sum: as with an integral, there may be a simple way to express the sum as another, sum-free function.
Even when we cannot find such a closed form, we may be able to *approximate* the value well enough for our purposes.

1 Annuities

Would you prefer a million dollars today or $50,000 a year for the rest of your life? This is a question about the value of an annuity.
An *annuity* is a financial instrument that pays out a fixed amount of money at the beginning of every year for some specified number of years.
In particular, an n-year, m-payment annuity pays m dollars at the start of each year for n years.
In some cases, n is finite, but not always.
Examples include lottery payouts, student loans, and home mortgages.
There are even Wall Street people who specialize in trading annuities.

A key question is what an annuity is worth.
For example, lotteries often pay out jackpots over many years.
Intuitively, $\$50,000$ a year for 20 years ought to be worth less than a million dollars right now.
If you had all the cash right away, you could invest it and begin collecting interest.
But what if the choice were between $\$50,000$ a year for 20 years and a *half* million dollars today?
Now it is not clear which option is better.

In order to answer such questions, we need to know what a dollar paid out in the future is worth today.
We will to assume that money can be invested at a fixed annual interest rate p.
These days a good estimate for p is around 8%; we'll use this value for the rest of the lecture.

Here is why the interest rate p matters.
Ten dollars invested today at interest rate p will become $(1+p) = \$10.80$ in a year, $(1+p)^2 \approx \$11.66$

in two years, and so forth.
Looked at another way, ten dollars paid out a year from now are only really worth $1/(1+p) \approx \$9.26$ today.
The reason is that if we had the \$9.26 today, we could invest it and would have \$10.00 in a year anyway.
Therefore, p determines the value of money paid out in the future.

1.1 The Value of an Annuity

Our goal is to determine the value of an n-year, m-payment annuity.
The first payment of m dollars is truly worth m dollars.
But the second payment a year later is worth only $m/(1+p)$ dollars.
Similarly, the third payment is worth $m/(1+p)^2$, and the n-th payment is worth only $m/(1+p)^{n-1}$.
The total value V of the annuity is equal to the sum of the payment values.
This gives:

$$V = \sum_{i=1}^{n} \frac{m}{(1+p)^{i-1}}$$

To compute the real value of the annuity, we need to evaluate this sum.
One way is to plug in m, n, and p, compute each term explicitly, and then add them up.
However, this sum has a special closed form that makes the job easier.
(The phrase "closed form" refers to a mathematical expression without any summation or product notation.
) First, lets make the summation prettier with some substitutions.

$$\begin{aligned} V &= \sum_{i=1}^{n} \frac{m}{(1+p)^{i-1}} \\ &= \sum_{j=0}^{n-1} \frac{m}{(1+p)^{j}} \quad \text{(substitute } j = i-1\text{)} \\ &= m \sum_{j=0}^{n-1} x^j \quad \text{(substitute } x = \frac{1}{1+p}\text{)} \end{aligned}$$

The goal of these substitutions is to put the summation into a special form so that we can bash it with a theorem given in the next section.

1.2 The Sum of a Geometric Series

Theorem 1.1 For all $n \geq 1$ and all $x \neq 0$,

$$\sum_{i=0}^{n-1} x^i = \frac{1-x^n}{1-x}.$$

The terms of the summation in this theorem form a *geometric series*.
The distinguishing feature of a geometric series is that each term is a constant times the one before; in this case, the constant is x.
The theorem gives a closed form for the sum of a geometric series that starts with 1.

Proof. The proof is by induction on n.
Let $P(n)$ be the predicate that for all $x \neq 1$, $\sum_{i=0}^{n-1} x^i = \frac{1-x^n}{1-x}$.
In the base case, $P(1)$ holds because $\sum_{i=0}^{0} x^i = x^0 = 1$ and $\frac{1-x^1}{1-x} = 1$.

In the inductive step, for $n \geq 1$ assume that for all $x \neq 1$, $\sum_{i=0}^{n-1} x^i = \frac{1-x^n}{1-x}$.
We will use this to prove that for all $x \neq 1$, $\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$.

$$
\begin{aligned}
\sum_{i=0}^{n} x^i &= x^n + \sum_{i=0}^{n-1} x^i \\
&= x^n + \frac{1-x^n}{1-x} \\
&= \frac{x^n(1-x) + 1 - x^n}{1-x} \\
&= \frac{1-x^{n+1}}{1-x}
\end{aligned}
$$

The second line follows from the first by the induction hypothesis.
The remaining steps are only simplifications.

∎

As if often the case, the proof by induction gives no hint about how the formula was found in the first place.
Here is a more insightful derivation.
The trick is to let S be the value of the sum:

$$
\begin{aligned}
S &= 1 + x + x^2 + \ldots + x^{n-1} \\
-xS &= -x - x^2 - x^3 - \ldots - x^n
\end{aligned}
$$

Adding these two equations gives:

$$
\begin{aligned}
(1-x)S &= 1 - x^n \\
S &= \frac{1-x^n}{1-x}
\end{aligned}
$$

We'll say more about finding (as opposed to proving) summation formulas later in the next lecture.

1.3 Return of the Annuity Problem

Now we can solve the annuity pricing problem.
The value of an annuity that pays m dollars at the start of each year for n years is computed as follows:

$$\begin{aligned} V &= m\sum_{j=0}^{n-1} x^j \\ &= m\frac{1-x^n}{1-x} \\ &= m\frac{1-(\frac{1}{1+p})^n}{1-\frac{1}{1+p}} \\ &= m\frac{1+p-(\frac{1}{1+p})^{n-1}}{p} \end{aligned}$$

The first line is a restatement of the summation we obtained earlier for the value of an annuity.
The second line follows by applying the theorem for the summation of a geometric series.
In the third line, we undo the earlier substitution $x = 1/(1+p)$.
In the final step, both the numerator and denominator are multiplied by $1+p$ to simplify the expression.

The resulting formula is much easier to use than a summation with dozens of terms.
For example, what is the real value of a winning lottery ticket that pays \$50,000 per year for 20 years? Plugging in $m = \$50,000$, $n = 20$, and $p = 0.8$ gives $V \approx \$530,180$.
Because payments are deferred, the million dollar lottery is really only worth about a half million dollars! This is a good trick for the lottery advertisers!

1.4 Infinite Geometric Series

The question at the beginning of this section was whether you would prefer a million dollars today or \$50,000 a year for the rest of your life.
Of course, this depends on how long you live, so optimistically assume that the second option is to receive \$50,000 a year *forever*.
This sounds like infinite money!

We can compute the value of an annuity with an infinite number of payments by taking the limit of our geometric sum in Theorem 1.1 as n tends to infinity.
This one is worth remembering!

Theorem 1.2 If $|x| < 1$, then

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}.$$

Proof.

$$\begin{aligned}\sum_{i=0}^{\infty} x^i &= \lim_{n\to\infty} \sum_{i=0}^{n-1} x^i \\ &= \lim_{n\to\infty} \frac{1-x^n}{1-x} \\ &= \frac{1}{1-x}\end{aligned}$$

The first equality follows from the definition of an infinite summation.
In the second line, we apply the formula for the sum of an n-term geometric series given in Theorem 1.1.
The final line follows by evaluating the limit; the x^n term vanishes since we assumed that $|x| < 1$. ■

In our annuity problem, $x = 1/(1+p) < 1$, so the theorem applies.
Substituting for x, we get an annuity value of

$$\begin{aligned}V &= m \cdot \frac{1}{1-x} \\ &= m \cdot \frac{1}{1-1/(1+p)} \\ &= m \cdot \frac{1+p}{(1+p)-1} \\ &= m \cdot \frac{1+p}{p}\end{aligned}$$

Plugging in $m = \$50,000$ and $p = 0.8$ gives only $\$675,000$.
Amazingly, a million dollars today is worth much more than $\$50,000$ paid every year forever! Then again, if we had a million dollars today in the bank earning 8% interest, we could take out and spend $\$80,000$ a year forever.
So the answer makes some sense.

1.5 Examples

We now have formulas enabling us to sum both finite and infinite geometric series.
Some examples are given below.
In each case, the solution follows immediately from either Theorem 1.1 (for finite series) or Theorem 1.2 (for infinite series).

$$\begin{aligned}
1 + 1/2 + 1/4 + 1/8 + \ldots &= \sum_{i=0}^{\infty} (1/2)^i \\
&= \frac{1}{1-(1/2)} \\
&= 2
\end{aligned}$$

$$\begin{aligned}
0.999999999\ldots &= 0.9 \sum_{i=0}^{\infty} (1/10)^i \\
&= 0.9 \frac{1}{1-1/10} \\
&= 0.9 \frac{10}{9} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
1 - 1/2 + 1/4 - 1/8 + \ldots &= \sum_{i=0}^{\infty} (-1/2)^i \\
&= \frac{1}{1-(-1/2)} \\
&= 2/3
\end{aligned}$$

$$\begin{aligned}
1 + 2 + 4 + 8 + \ldots + 2^{n-1} &= \sum_{i=0}^{n-1} 2^i \\
&= \frac{1-2^n}{1-2} \\
&= 2^n - 1
\end{aligned}$$

$$\begin{aligned}
1 + 3 + 9 + 27 + \ldots + 3^{n-1} &= \sum_{i=0}^{n-1} 3^i \\
&= \frac{1-3^n}{1-3} \\
&= \frac{3^n-1}{2}
\end{aligned}$$

If the terms in a geometric series grow smaller as in the first example, then the series is said to be *geometrically decreasing*.
If the terms in a geometric series grow progressively larger as in the last two examples, then the series is said to be *geometrically increasing*.

Here is a good rule of thumb: *the sum of a geometric series is approximately equal to the term with greatest absolute value*.
In the first two examples, the largest term is equal to 1 and the sums are 2 and 2/3, both relatively close to 1.
In the third example, the sum is about twice the largest term.

In the final example, the largest term is 3^{n-1} and the sum is $(3^n - 1)/2$, which is only about a factor of 1.5 greater.

1.6 Related Sums

We now know all about sums of geometric series.
But in practice one often encounters sums that cannot be transformed by simple variable substitutions to the form $\sum x^i$.

A non-obvious, but useful way to obtain new summation formulas from old is by differentiating or integrating with respect to x.
As an example, consider the following series.

$$\sum_{i=1}^{n} ix^i = x + 2x^2 + 3x^3 + \ldots + nx^n$$

This is not a geometric series, since the ratio between successive terms is not constant.
Our formula for the sum of a geometric series cannot be directly applied.
But suppose that we differentiate that formula:

$$\begin{aligned}
\frac{d}{dx}\sum_{i=0}^{n} x^i &= \frac{d}{dx}\,\frac{1-x^{n+1}}{1-x} \\
\sum_{i=0}^{n} ix^{i-1} &= \frac{-(n+1)x^n(1-x) - (-1)(1-x^{n+1})}{(1-x)^2} \\
&= \frac{-(n+1)x^n + (n+1)x^{n+1} + 1 - x^{n+1}}{(1-x)^2} \\
&= \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}
\end{aligned}$$

Often differentiating or integrating messes up the exponent of x in every term.
In this case, we now have a formula for a series of the form $\sum ix^{i-1}$, but we want a formula for the series $\sum ix^i$.
The solution is simple: multiply by x.
This gives:

$$\sum_{i=0}^{n} ix^i = \frac{x-(n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$$

Since we could easily have made a mistake, it is a good idea to go back and validate a formula obtained this way with a proof by induction.

Notice that if $|x| < 1$, then this sum converges to a finite value even if there are infinitely many terms.
Taking the limit as n tends infinity gives the following theorem:

Theorem 1.3 If $|x| < 1$, then

$$\sum_{i=0}^{\infty} ix^i = \frac{x}{(1-x)^2}.$$

As a consequence, suppose there is an annuity that pays im dollars at the *end* of each year i forever. For example, if $m = \$50,000$, then the payouts are $\$50,000$ and then $\$100,000$ and then $\$150,000$ and so on.
It is hard to believe that the value of this annuity is finite! But we can use the preceding theorem to compute the value:

$$\begin{aligned} V &= \sum_{i=1}^{n} \frac{m}{(1+p)^i} \\ &= m\frac{\frac{1}{1+p}}{(1-\frac{1}{1+p})^2} \\ &= m\frac{1+p}{p^2} \end{aligned}$$

The second line follows by an application of Theorem 1.3.
The third line is obtained by multiplying the numerator and denominator by $(1+p)^2$.

For example, if $m = \$50,000$, and $p = 0.08$ as usual, then the value of the annuity is $V = \$8,437,500$.
Even though payments increase every year, the increase is only additive with time; by contrast, dollars paid out in the future decrease in value exponentially with time.
The geometric decrease swamps out the additive increase.
Payments in the distant future are almost worthless, so the value of the annuity is finite.

The important thing to remember is the trick of taking the derivative (or integral) of a summation formula.
Of course, this technique requires one to compute nasty derivatives correctly, but this is at least theoretically possible!

2 Book Stacking

Suppose n books are stacked on the edge of a table as shown in Figure 1.
How far can the top book overhang the table edge without the whole stack falling over? One book length? Two? We will prove that if there are enough books, then the top book can overhang arbitrarily far!

Figure 1: *If n books are stacked on a table, how far can the top book overhang the edge without the whole stack falling over?*

2.1 Formalizing the Problem

First, we have to reduce the real-world problem to a mathematical question.
Define the length of a book to be 2 and assume that all books have the same mass.
For $0 \leq i \leq n-1$, define d_i to be the distance that the top book's (book 0's) right edge extends beyond book i's right edge.
Treat the table as book n; that is, d_n is the distance book 0 extends beyond the table.
Note that while intuitively the d_i should be decreasing (each book extends farther than the one below it) there may be exceptions as shown by book 3 in the figure.

Figure 2: *For $1 \leq i \leq n$, the variable d_i is the distance that book 0 extends beyond the right end of book i. The table is regarded as book $n+1$, so x_n is the distance that book 1 extends beyond the end of the table. The d_i need not decrease monotonically, as shown by d_3.*

The position of the books is constrained by the condition that the stack must not fall over.
A stack of books is *stable* iff for $1 \leq k \leq n$, the center of mass for the top k books lies above the $(k+1)$-st book.
We will not prove this, but will accept it as an principle of physics (*i.e.*, an axiom).
This stability constraint is illustrated for $n = 1$ in Figure 3.

Let's apply a greedy strategy to our book stacking problem.

Figure 3: *This figure illustrates the idea of a stable stack of books for the special case of a single book. In the left diagram, the stack is stable because the center of mass lies above the next "book" (in this case, the table). In the right diagram, the book is unstable because the center of mass overhangs.*

Put the top book so that its center of mass is at the edge of the second book, and then place the second book so that the center of mass of the top two is at the edge of the third book, and so on. In general, we place the kth book such that the center of mass of the top k books is at the edge of the $(k+1)$st book.
That is, we want d_k to be equal to the center of mass of the top k books.
Let's write the equation for this.

As we do these equations, we are working "relative to the right side of the top book"—our coordinate frame measures distance to the **left** of this point.

Notice that the center of mass of the book number i is at distance d_i+1 to the left of the top book. So the center of mass of the top k books (numbered $0,\ldots,k-1$) is at the average of these k positions, namely

$$\frac{(d_{k-1}+1)+(d_{k-2}+1)+\cdots+(d_0+1)}{k}$$

and we want this to be just above the right hand side of book number k, which is at offset d_k. This gives us an equation relating the quantities:

$$d_k = \frac{(d_{k-1}+1)+(d_{k-2}+1)+\cdots+(d_0+1)}{k}$$

which rewrites as

$$kd_k = d_{k-1}+d_{k-2}+\cdots+d_0+k$$

Of course, by the same reasoning,

$$(k-1)d_{k-1} = d_{k-2}+d_{k-3}+\cdots+d_0+k-1$$

and if we subtract this equation from the one before, we get

$$kd_k-(k-1)d_{k-1} = d_{k-1}+1$$

or in other words,

$$d_k = d_{k-1} + 1/k$$

Now if we "expand" the relation above, we find out that

$$\begin{aligned} d_k &= d_{k-1} + 1/k \\ &= d_{k-2} + 1/(k-1) + 1/k \\ &\qquad\qquad\qquad \cdots \\ &= 1 + 1/2 + \cdots + 1/(k-1) + 1/k \end{aligned}$$

Figure 4: *This is a stacking of 4 books that overhangs the table by the greatest possible distance. In general, the k-th book from the top should extend distance $\frac{1}{k}$ beyond the book below.*

Can we do better? Well, for stability, looking at the derived center of mass above, we *must* have

$$d_k \leq \frac{(d_{k-1} + 1) + (d_{k-2} + 1) + \cdots + (d_0 + 1)}{k}$$

and in our greedy strategy above, the inequality is always "tight" (met with equality): that is, d_k is as large as possible, given $d_0, \ldots, d_{k-1}$.
It follows by induction that each d_k is as large as possible.
So we cannot do better.

2.2 Evaluating the Sum– The Integral Method

The best possible total extension is now expressed as a sum that, coincidentally, appears frequently in computer science.
(Fancy that!) In fact, it is important enough to have a name.

Definition The *n-th Harmonic number* is equal to $\sum_{i=1}^{n} \frac{1}{i}$ and is denoted H_n.

We can rewrite the formula for greatest possible extension simply as $H_n/2$.

The first few Harmonic numbers are easy to compute.
For example, $H_1 = 1$, $H_2 = 1 + \frac{1}{2} = \frac{3}{2}$, $H_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{16}$, $H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$.
The fact that H_4 is greater than 2 has special significance; it implies that the total extension of a 4-book stack is greater than one full book! This is the situation shown in Figure 4.

It would be nice to answer questions like, "How many books are needed to build a stack extending 100 book lengths beyond the table?" One approach to this question would be to keep computing Harmonic numbers until we found one exceeding 200.
However, as we will see, this is not such a keen idea.
In fact, for all we know, building a stack extending so far may be impossible regardless of the number of books.

All such questions would be settled if we could express H_n in a closed form.
Unfortunately, no closed form is known and probably none exists.
As a second best, however, we can find closed forms for very good approximations to H_n using the Integral Method.
The idea of the Integral Method is to bound terms of the sum above and below by simple functions as suggested in Figure 5.
The integrals of these functions then bound the value of the sum above and below.

Figure 5: *This figure illustrates the Integral Method for bounding a sum. The area under the "stairstep" curve over the interval* $[0, n]$ *is equal to* $H_n = \sum_{i=1}^{n} 1/i$*. The function* $1/x$ *is everywhere greater than or equal to the stairstep and so the integral of* $1/x$ *over this interval is an upper bound on the sum. Similarly,* $1/(x+1)$ *is everywhere less than or equal to the stairstep and so the integral of* $1/(x+1)$ *is a lower bound on the sum.*

The Integral Method gives the following upper and lower bounds on the harmonic number H_n:

$$
\begin{aligned}
H_n &\leq 1 + \int_1^n \frac{1}{x}\,dx = 1 + \ln n \\
H_n &\geq \int_0^n \frac{1}{x+1}\,dx = \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1)
\end{aligned}
$$

These bounds imply that the harmonic number H_n is around $\ln n$.
Since $\ln n$ grows without bound, albeit slowly, we can make a stack of books that extends arbitrarily

far.

For example, to build a stack extending 100 book lengths beyond the table, we need a number of books n so that $H_n = 200$.
Exponentiating the above inequalities gives:

$$\begin{aligned} e^{H_n - 1} \leq \quad & n \quad \leq e^{H_n} - 1 \\ e^{199} \leq \quad & n \quad \leq e^{200} - 1 \end{aligned}$$

The number of books required has 86 digits!

2.3 More about Harmonic Numbers

In the preceding section, we showed that H_n is about $\ln n$.
A even better approximation is known:

$$H_n = \ln n + \gamma + \frac{1}{2n} + \frac{1}{12n^2} + \frac{\epsilon(n)}{120n^4}$$

Here γ is a value $0.577215664\ldots$ called Euler's constant, and $\epsilon(n)$ is between 0 and 1 for all n.
We will not prove this formula.

The shorthand $H_n \sim \ln n$ is used to indicate that the leading term of H_n is $\ln n$.
More generally, the notation $f(n) \sim g(n)$ means that $\lim_{n\to\infty} \frac{f(n)}{g(n)} \to 1$.

We also might write $H_n \sim \ln n + \gamma$ to indicate two leading terms.
While this notation is widely used, it is not really right.
Referring to the definition of $\sim$, we see that while $H_n \sim \ln n + \gamma$ is a true statement, so is $H_n \sim \ln n + c$ where c is any constant.
The correct way to indicate that γ is the second-largest term is $H_n - \ln n \sim \gamma$.

The reason that the $\sim$ notation is useful is that often we do not care about lower order terms.
For example, if $n = 100$, then we can compute $H(n)$ to great precision using only the two leading terms:

$$|H_n - \ln n - \gamma| \leq |\frac{1}{200} - \frac{1}{120000} + \frac{1}{120 \cdot 100^4}| < \frac{1}{200}$$