# Problem Set 2 Solutions

**Problem 1.** Suppose you take a piece of paper and draw a bunch of straight lines, each in its own arbitrary direction, that completely cross the paper. This divides the paper up into polygonal regions. Prove by induction that you can always color the various regions using only *two* colors, so that any two regions that share a boundary line are different colors. Note that this need not be true of regions that share only a boundary point.

**Solution.**

**Solution 1:** For this solution, we assume that all lines are straight and completely cross the paper. Also, when we say that the regions are colored, we always mean they are colored using only [[[at most?]]] two colors, red and green. [[[Is it better to just say "using two colors" everywhere?]]] We say that two regions are *adjacent* if they share a boundary line. Note that two regions cannot share two distinct boundary lines.

Let $P(n)$ be the proposition that the regions formed by any set of $n$ distinct lines can be colored so that (two) adjacent regions are different colors. We prove $\forall n \in \mathbb{N}\ P(n)$ by induction (on $n$).

**Base case:** $P(0)$**.** There is only one region, and thus there are no adjacent regions. So this case is trivially true. The region may be colored either red or green.

**Inductive step:** $\forall n \in \mathbb{N}\ (P(n) \implies P(n+1))$**.** We must show that if the regions formed by any $n$ distinct lines can be colored so that adjacent regions have different colors, then the regions formed by any $n+1$ distinct lines can be colored so that adjacent regions have different colors.

1. Fix $n \in \mathbb{N}$.
2. Assume that the regions formed by any $n$ distinct lines can be colored so that adjacent regions have different colors.
3. The regions formed by any $n+1$ distinct lines can be colored so that adjacent regions have different colors.
   (a) Fix a set of $n+1$ lines. These lines divide the paper into a set $X$ of regions.
   (b) Select any line $l$ from that set. There are $n$ other lines, which divide the paper into a set $Y$ of regions, such that every region in $X$ is either a region in $Y$ or else it is part of a region in $Y$ that is split by $l$.
   (c) By the inductive hypothesis, the regions in $Y$ can be colored so adjacent regions are different colors. This also gives a coloring for the regions in $X$. Call this coloring of the regions in $X$ the "original coloring."
   (d) Line $l$ splits the paper into two sides. Select one of the sides and flip all the colors of the regions on that side. Call this coloring of the regions in $X$ the "new coloring." We prove that, using the new coloring, any adjacent regions (in $X$) have different colors.
      i. Fix any two adjacent regions in $X$.

**Case 1:** They share boundary line $l$. The two regions had the same color in the original coloring. The two regions are on opposite sides of $l$. Therefore, the color in exactly one of the two regions was flipped, and they have different colors in the new coloring.

**Case 2:** They do not share boundary line $l$. The two regions had different colors in the original coloring, and they are on the same side of $l$. Thus, the color of either none or both regions was flipped, and the two regions have different colors in the new coloring.

ii. QED (universal generalization over adjacent regions)

(e) QED (universal generalization over sets of $n+1$ lines)

4. QED (implication)

QED (induction)

**Solution 2:** We say a piece of paper divided into regions is 2-colorable if the regions can be colored using only two colors so that any two regions that share a boundary line are different colors.

Let $P(n)$ be the proposition that a piece of paper with any $n$ straight, completely-crossing lines is 2-colorable. We want to prove, by induction on the number of lines, $\forall n \in \mathbb{N}\ P(n)$.

(For the purposes of this proof, we assume all lines are distinct, straight, and completely cross the paper.)

1. (Base case: Prove $P(0)$.) Select one of your colors; color the entire piece of paper with it. Note that there are no two regions that share a boundary line therefore this satisfies the condition trivially.
2. (Inductive case: Prove $\forall n \in \mathbb{N}\ (P(n) \implies P(n+1))$.) Substituting our definition of $P(n)$, we must show that if a paper divided up by any $n$ lines is 2-colorable then a paper divided up by $n+1$ lines is 2-colorable, for all natural numbers $n$.
   (a) Fix $n \geq 0$.
   (b) Assume a piece of paper divided up by any $n$ lines is 2-colorable.
   (c) A piece of paper divided by $n+1$ lines is 2-colorable.
      i. Fix a piece of paper with $n+1$ lines.
      ii. Select any line $l$ on the page. Consider the piece of paper as divided by the other $n$ lines. By the inductive hypothesis, the $n$-line page is 2-colorable.
      iii. Line $l$ divides the paper into two sides. Select one of the sides and flip the colors of all the regions on that side. The piece of paper with $n+1$ lines is now 2-colored.
         A. Consider any two regions that share a boundary line. They are different colors.

            **Case 1:** The boundary line is part of $l$. The two regions had the same color on the $n$-line page. The two regions are on opposite sides of $l$. Therefore, the color in exactly one of the two regions was flipped.

            **Case 2:** The boundary line is not part of $l$. The two regions had different colors on the $n$-line page. The two regions are on the same side of $l$. Therefore, the color of either none or both regions was flipped.

            QED (cases)
         B. QED (universal generalization over the regions)
      iv. QED (universal generalization over pieces of paper with $n+1$ lines)

   (d) QED (Implication)

3. QED (Induction)

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**Problem 2.** Prove by induction that $\forall n \geq 1$, the quantity $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

**Solution.** Let $P(n)$ be the predicate that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9, *i.e.*, $P(n) = 9|(n^3 + (n+1)^3 + (n+2)^3)$. We prove that $P(n)$ holds for all $n \geq 1$ by induction on $n$.

**Base Case:**$P(1)$
  $P(1) = 9|(1^3 + 2^3 + 3^3) = 9|36 = \texttt{True}$
**Inductive Step:**$\forall n \geq 1, P(n) \implies P(n+1)$

1. Fix $n \geq 1$.
2. From the inductive hypothesis $P(n)$ holds, *i.e.*, $9|(n^3 + (n+1)^3 + (n+2)^3)$.
3. We must show that $P(n+1)$ holds, *i.e.*, $9|((n+1)^3 + (n+2)^3 + (n+3)^3)$.
   (a) $(n+1)^3 + (n+2)^3 + (n+3)^3 = n^3 + (n+1)^3 + (n+2)^3 + 9(n^2 + 3n + 3)$
   (b) Must show: $9|(n^3 + (n+1)^3 + (n+2)^3 + 9(n^2 + 3n + 3))$.
   (c) From the induction hypothesis, $9|(n^3 + (n+1)^3 + (n+2)^3)$.
   (d) Also, it is simple to see that $9|(9(n^2 + 3n + 3))$.
   (e) Since 9 divided both terms, it divides their sum.
   (f) QED
4. QED (implication and UG)

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**Problem 3.** Prove by induction on $n$ that for all $n \geq 0$, the inequality $|x| + |y| \leq n$ has exactly $2n^2 + 2n + 1$ solutions where $x$ and $y$ are integers.

**Solution.** The proof is by induction on $n$. Let $P(n)$ be the proposition that there are $2n^2 + 2n + 1$ solutions to the inequality $|x| + |y| \leq n$, where $x$ and $y$ are integers. In the base case, $P(0)$ is true because $2(0)^2 + 2(0) + 1 = 1$, and there is one solution to the equation $|x| + |y| \leq 0$, namely $x = y = 0$.

In the inductive step, assume $P(n)$ to prove $P(n+1)$. That is, we want to prove that there are $2(n+1)^2 + 2(n+1) + 1 = 2n^2 + 6n + 5$ solutions to the inequality $|x| + |y| \leq n+1$. Every solution to this inequality is also either a solution to the tighter inequality $|x| + |y| \leq n$ or else to the equation $|x| + |y| = n+1$. Thus, the number of solutions is equal to the number of solutions to the tighter inequality plus the number of solutions to the equation.

By induction, there are $2n^2 + 2n + 1$ solutions to the tighter inequality. All that remains is to count solutions to the equation. The value of $x$ must be in the range $-(n+1), \ldots, n+1$. If $x$ takes on either of the two values $-(n+1)$ or $n+1$, then there is a single solution, with $y = 0$. If

$x$ takes on one of the $2n+1$ values in the range $-n, \ldots, n$, then there are two solutions, one with $y = n+1-|x|$ and one with $y = -(n+1-|x|)$. Thus, there are $2 + 2 \cdot (2n+1) = 4n+4$ solutions to the equation and $2n^2+2n+1$ solutions to the tighter inequality, giving $2n^2+6n+5$ solutions in all, as desired. Therefore, $P(n)$ implies $P(n+1)$, and the claim is proved by induction.

$\square$

**Problem 4.** The 6.042 mascot, Math Moose, made an amazing observation while playing with his collection of unit squares over the weekend.

First, Math Moose put his favorite unit square down on the floor as in diagram (a) of Figure 1. He noticed that the length of the periphery of the resulting shape was 4, an even number. Next, he put a second unit square down next to the first so that the two squares shared an edge as in diagram (b). He noticed that the length of the periphery of the resulting shape was now 6, which is also an even number. (The periphery of each shape in the figure is indicated by a thicker line.) Math Moose continued to place squares so that each new square shared an edge with at least one previously-placed square and no squares overlapped. Eventually, he arrived at the shape in diagram (c). He realized that the length of the periphery of this shape was 36, which is again an even number.

Math Moose spent the rest of the weekend trying to create a shape with an odd-length periphery, without any success. Help him out by proving by induction on the number of squares that the periphery length is always even, no matter how many squares he places or how he arranges them.

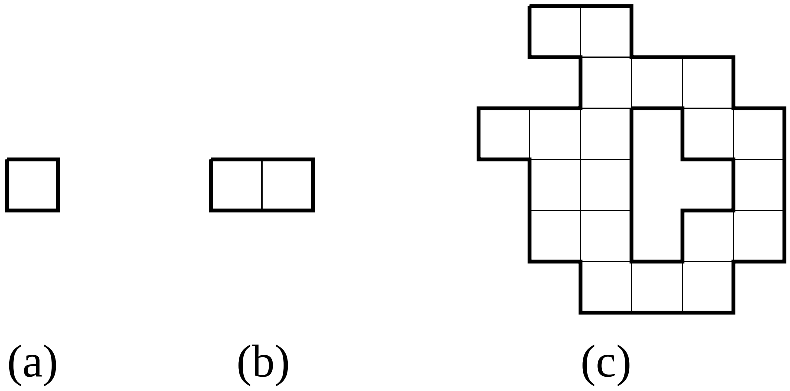



**Figure 1**: Some of the shapes that Math Moose created.

**Solution.** Let $P(n)$ be the predicate that the periphery length is even after $n$ squares are placed. We prove $P(n)$ for all $n \geq 1$ by induction on $n$.

In the base case, $P(1)$ is true because the periphery of a single square has length 4, which is even.

In the inductive step, assume that the periphery length is even after $n$ squares are placed to prove that the periphery length is even after $n+1$ squares are placed. The $(n+1)$-st square could share 1, 2, 3, or 4 edges with previously-placed squares.

- If the new square shares 1 edge with a previously placed square, then this one edge is removed from the periphery, but three edges of the new square are added to the periphery. Overall, the periphery length increases by two and thus remains even.

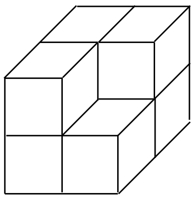
- If the new square shares 2 edges with previously placed squares, then these two edges are removed from the periphery, but two edges of the new square are added. The periphery length is unchanged and thus remains even.
- If the new square shares 3 edges, then these three edges are removed from the periphery, but one edge is added. The periphery length decreases by two and remains even.
- If the new square shares 4 edges, then these four edges are removed from the periphery and none are added. The periphery length decreases by four and remains even.

In all cases, the length of the periphery remains even. Therefore, for all $n \geq 1$, $P(n)$ implies $P(n+1)$.

□

**Problem 5.** The C.I.A. wants you to design plans to build, for an arbitrary $k \geq 1$, a $2^k \times 2^k \times 2^k$ cubic foot granite cube that is solid except for a missing single cubic foot that is one of the eight cubic feet nearest the center of the cube. It will have one of its corners exactly at the center of the large cube. The C.I.A. is not telling you what is to be placed inside the cube.

You are to build this by cementing together *basic blocks* that are 2-foot $\times$ 2-foot $\times$ 2-foot cubes, with one one-foot cube missing from its corner — as shown in the diagram below.



Describe your method for doing this. Your plans should be described recursively on $k$.

**Solution.** First, we prove the following theorem:

**Theorem.** *Let $n$ be a positive integer. Show that a $2^n \times 2^n \times 2^n$ cube, (an $n$-*cube*) with one arbitrary cubic foot removed can be built with basic blocks of size $2 \times 2 \times 2$ (1-*cube*) with one of its cubic foot missing.*

**Proof.** Let $P(n)$ be the statement that any $n$-*cube* with one arbitrary cubic foot missing can be built using 1-*cubes* with one cubic foot removed. We use mathematical induction to prove $P(n)$ is true for all positive integers $n$.

**Basis Step** $P(1)$ is true since a 1-*cube* with one missing cubic foot is already provided as basic block by CIA.

**Inductive Step:** Assume that $P(n)$ is true. We show that $P(n+1)$ is true. Construct 8 $n$-*cubes* in such a way that 7 of them are missing one cubic foot from one of their 8 corners, and the 8*th* one is missing an arbitrary cubic foot. We can do this by the induction hypothesis. We construct an $(n+1)$-*cube* as follows: Put the 7 corner-missing $n$-*cubes* together so that all of their missing corners meet in the center. Then fill up the gap with a basic block. Now add the 8*th* $n$-*cube* with an arbitrary cubic foot missing to get an $(n+1)$-*cube* with an arbitrary cubic foot removed. ■

Now we are in good shape to prove the master theorem.

**Theorem.** *Prove that an* $(n+1)$-cube *with one cubic foot missing from its center (the* **CIA** *cube) can be built using* 1-cubes *with one cubic foot missing.*

**Proof.** The **CIA** cube can be built by following the steps in the proof of **Theorem** 0.1 except that when building the 8*th* cube, we choose the arbitrary missing cubic foot to be one of the 8 possible 1-*cubes* in the center of the 8*th* $n$-*cube*. ■

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**Problem 6.** One plan considered by the 6.042 staff, then dropped, involved requiring students in tutorials to work in groups of either exactly 4 or exactly 7 students. We abandoned this plan when we realized that a tutorial section of size 13 could not be split up in this way.

(a) What is the smallest number $k$ such that all tutorials of $k$ or more students *can* be split into teams of 4 and 7?

(b) Prove that $k$ is the smallest such number.

(c) Prove (by induction) that all tutorials of $k$ or more students can be split into teams of size 4 and 7.

**Solution.**

**(a)** The smallest number $k$ such that all tutorials of $k$ or more students *can* be split into teams of 4 and 7 is 18.

**(b)** A tutorial of size 17 can not be split into teams of 4 and 7. In particular, a tutorial of size 17 can contain at most 2 groups of 7. Consider the various possible splits of the tutorial: i) no groups of 7: since the only other group size is 4 we can not split the tutorial of 17; ii) one group of 7: the remaining 10 students can not be split into groups of 4; iii) two groups of 7: the remaining 3 students can not be split into groups of 7.

**(c)** The proof is by strong induction on the size $n$ of the tutorial. Let $P(n)$ be the predicate that a tutorial of size $n$ can be split into groups of size 4 and 7.
We must show that $P(n)$ holds for all $n \geq 18$.

1. **Base Case:** $P(18)$, $18 = 4 + 7 + 7$
2. **Base Case:** $P(19)$, $19 = 4 + 4 + 4 + 7$
3. **Base Case:** $P(20)$, $20 = 4 + 4 + 4 + 4 + 4$
4. **Base Case:** $P(21)$, $21 = 7 + 7 + 7$
4. **Inductive Step:** $\forall n \geq 21, P(n) \implies P(n+1)$
    1. Fix $n \geq 21$.
    2. From the inductive hypothesis $P(x)$ holds, for all $18 \leq x \leq n$.
    3. In particular, $P(n-3)$ holds.
    3. The size of the $n+1$ student tutorial is 4 students greater than the $n-3$ student tutorial.
    4. We can split up the $n+1$ tutorial as the $n-3$ tutorial
    and with an extra 4 student group.

5. QED

3. QED

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**Problem 7.** Let the sequence $G_0, G_1, G_2, \ldots$ be defined recursively as follows: $G_0 = 0$, $G_1 = 1$, and $G_n = 5G_{n-1} - 6G_{n-2}$, for every $n \in \mathbb{N}, n \geq 2$.

Prove that for all $n \in \mathbb{N}$, $G_n = 3^n - 2^n$.

**Solution.** Let $P(n)$ be the predicate that $G_n = 3^n - 2^n$. We prove $P(n)$, for all $n \in \mathbb{N}$, by strong induction on $n$.

**Base Case:**

In order to simplify the strong induction, we use two base cases, in particular $n = 0$ and $n = 1$. By the definition of the sequence it is the case that $G_0 = 0$ (and $G_1 = 1$). Moreover, since $n = 0$ (and $n = 1$) it is the case that $3^n - 2^n = 3^0 - 2^0 = 1 - 1 = 0$ (and $3^n - 2^n = 3^1 - 2^1 = 3 - 2 = 1$). Thus it follows that $P(0)$ (and $P(1)$).

**Inductive Step:**

We must show that for all $n \in \mathbb{N}, n \geq 2$, it is the case that $P(n) \implies P(n+1)$.

1. Fix $n \in \mathbb{N}, n \geq 2$.
2. Assume that the elements of the sequence $G_x$, for $0 \leq x \leq n$, can be expressed as $G_x = 3^x - 2^x$.
3. The element of the sequence $G_{n+1}$ can be expressed as $G_{n+1} = 3^{n+1} - 2^{n+1}$.
   (a) By the definition of the sequence we have $G_{n+1} = 5G_n - 6G_{n-1}$.
   (b) By our induction hypothesis, it is the case that $G_n = 3^n - 2^n$ and $G_{n-1} = 3^{n-1} - 2^{n-1}$
   (c) Thus, $G_{n+1} = 5(3^n - 2^n) - 6(3^{n-1} - 2^{n-1})$. Working through the algebra, we conclude that $G_{n+1} = 3^{n+1} - 2^{n+1}$, as needed for the inductive step.
   (d) QED
4. QED (implication and UG)

QED (induction)

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**Problem 8.** Construct the following rankings for $n$-player tournaments, for any $n \geq 2$, or show that this is not always possible.

**(a)** Assume that one player, Enrique, wins all his matches, and another, Ricky, wins all but one of his matches. Construct a ranking that is locally fair, and in which Enrique is first and Ricky second.

**(b)** Assume that there are no sets of 3 players, A, B, and C, such that A beats B, B beats C, and C beats A; that is, $\nexists A, B, C, [(A \text{ beats } B) \wedge (B \text{ beats } C) \wedge (C \text{ beats } A)]$. Construct a ranking that is extremely fair, in the sense that any player X that beats another player Y appears higher than Y in the rankings.

**Solution.**

**(a)** Since Enrique beat everyone and Ricky beat everyone but one person, it follows that Ricky lost to Enrique. Now, consider all the other players in the tournament. As seen in class, they can be ordered in a locally fair ranking. It follows that a locally fair ranking for all the players is the ranking with Enrique first, Ricky second, and the remaining players ranked based on a locally fair ranking. We know that Enrique beat Ricky, that Ricky beat everyone except Enrique, and that the remaining players can be ranked in a locally fair fashion.

**(b)** First, consider the case when there are only two players. In this case, an extremely fair ranking can be constructed by placing the winner first and the loser second. Now consider a tournament of at least 3 players, in which there are no 3-player cycles. In class we have shown that if a tournament has a cycle, then the tournament has a 3-player cycle. Thus, the contrapositive is also true; that is, if a tournament has no 3-player cycle, then the tournament has no cycles. We claim, that in a tournament with no cycles, a locally fair ranking is also extremely fair.

Consider a locally fair ranking and suppose that this ranking were not extremely fair. Since the ranking is not extremely fair, there exist players $X$ and $Y$ such that $X$ appears higher in the ranking than $Y$ while $Y$ beat $X$, *i.e.*, $\cdots X \to \cdots \to Y \cdots$, for and $Y \to X$. It follows that the tournament graph contains the cycle $X \to \cdots \to Y \to X$. Since it contains a cycle, it must contain a 3-player cycle. This contradicts the fact that the tournament has no 3-player cycles; that is, our assumption that the ranking is not extremely fair is false.

$\square$

**Problem 9.** The set $W$ of "well-formed parenthesis strings" can be defined by the following rules:

1. the empty string is in $W$, and
2. if $s$ and $t$ are in W then the string $(s)t$ is in $W$.

For example, the string $(()())()$ is a well-formed parenthesis string, where as the string $)(())$, isn't.

Use structural induction to prove that any prefix of any well-formed parenthesis string has at least as many left parentheses as right parentheses.

**Solution.** Let $P(t)$, for $t \in W$, be the predicate that any prefix of a well-formed parenthesis string $t$ has at least as many left as right parentheses. We prove that $P(t)$ holds for all $t \in W$ by structural induction.

**Base Case:**

Let $t$ be the empty string $\lambda$. $P(\lambda)$ holds since there are no left or right parentheses.

**Inductive Step:**

We must show that for all $s, t \in W$, $P(s)$ and $P(t)$ imply $P((s)t)$. Let $s, t$ be any strings in $W$ and assume that $P(s)$ and $P(t)$ hold. We must show that $P((s)t)$ holds; that is, any prefix of the string $(s)t$ has at least as many left as right parentheses. For any $s_1$ and $s_2$ let $s_1 \leq s_2$ denote that $s_1$ is a prefix of the string $s_2$. Let $p$ be any prefix of the string $(s)t$ and consider the various such cases:

$p =$ (: In this case, $p$ the claim is satisfied.

$p = (s'$, for some string $s'$, $s' \leq s$: In this case, since $P(s)$ holds, it follows that $s'$ has at least as many left as right parentheses. Thus, the same property holds for the string $(s'$.

$p = (s)$: In this case, since $P(s)$ holds, it follows that $(s)$ has at least as many left as right parentheses.

$p = (s)t'$, for some string $t'$, $t' \leq t$: In this case, by the previous case, we know that the string $(s)$ has at least as many left as right parentheses. Moreover, since $P(t)$ holds, it follows that $t'$ has at least as many left as right parentheses. Thus, it follows that the string $(s)t'$ has as many left as right parentheses.

Thus, any prefix of $(s)t$ has at least as many left as right parentheses; that is, $P((s)t)$ holds. Finally, by existential generalization, the inductive step holds any such $s, t \in W$.

$\square$