Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
Professors David Karger and Nancy Lynch

Handout 7
February 11, 2000

Mini-Quiz 2

1. Write your name:

2. (Rosen, Sec. 1.5, Ex. 21) Let A and B be subsets of a universal set U. Show that $A \subseteq B$ if and only if $\overline{B} \subseteq \overline{A}$.

3. (Rosen, Sec. 1.6, Ex. 15) Determine whether each of the following functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

   (a) $f(x) = 2x + 1$
   (b) $f(x) = x^2 + 1$
   (c) $f(x) = x^3$

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Tutorial 2 Problems

Problem 1. False Proofs

(a) Consider the following recursive definition of the set F of full binary trees:

1. A single node is in F.
2. If t and t' are in F, then the structure formed from t, t', and a single new node, with an arrow labeled "left" from the new node to t and another arrow labeled "right" from the new node to t', is also in F.

The recursive definition of the set F of full binary trees is similar to the one for binary trees, except that we only allow the recursive definition step that includes both a left and a right child.

Find the flaw in the following proof. Define the *height* of a binary tree to be 0 for a single node, otherwise one more than the larger of its two children's heights.

Claim. *Any height-n binary tree has exactly $2^n - 1$ nodes.*

Proof. By induction. The base case is clear. For the inductive step, assume it is true for height-n binary trees. Putting two of them together under a new node gives a height $n + 1$ binary tree which has $2 * (2^n - 1) + 1 = 2^{n+1} - 1$ nodes. This proves the inductive step. ■

(b) Claim. *For $n \in \mathbb{N}, n \geq 1$, any $2n \times 2n$ square grid can be tiled by L-shaped tiles such that any particular square is left untiled.*

Proof. The proof is by strong induction on n. Let $P(n)$ be the predicate: "Any $2n \times 2n$ square grid can be tiled by L-shaped tiles such that any particular square is left untiled".

Base Case:

The base case, for $n = 1$, is very simple, just use one tile.

Inductive Step:

Assume that the claim is true for all square grids smaller than $2n$, and we prove it for the $2n \times 2n$ square grid. This follows the proof of the $2^n \times 2^n$ case: Given a $2n \times 2n$ square grid, with one square designated to be left untiled, we construct a tiling (using L-shaped tiles) for all the other squares, as follows: Divide the square grid up into four quadrants, each $n \times n$. By the inductive hypothesis (since $n < 2n$ for all $n \geq 1$), we can tile each of these quadrants (with the L-shaped tiles) completely, except for one square of our choice. For the quadrant with the square to be left untiled, choose that square. For the others, choose the squares at the corner where the quadrants meet. This forms an L-shaped space at the center of the $2n \times 2n$ square grid, which can be filled by a single L-shaped tile. Thus, the entire $2n \times 2n$ square grid is tiled, except for the square designated.

■

Point out any logical errors in this proof (that is, not simply places where some steps were skipped, but where there is actually an error, though you are free to point out places where more steps should have been included for clarity). Then, either construct a correct proof, or show that the claim is false.

Problem 2. Let F_n be the nth Fibonacci number, for all $n \in \mathbb{N}$. Recall that the Fibonacci numbers are defined recursively as follows: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, for all $n \in \mathbb{N}, n \geq 2$.

(a) Prove that for all $n \in \mathbb{N}$, $\sum_{i=0}^{n} F_i = F_{n+2} - 1$.

(b) Find a formula for $\sum_{i=0}^{n} F_{2i}$ and prove that the formula is correct.

Problem 3.

(a) Give the structural inductive definition of the set F of full binary trees.

(b) Define the function $numleaves(t)$, for $t \in F$, giving the number of "leaves" of the full binary tree t; that is, $numleaves(t)$, for $t \in F$, gives the number of nodes in t that do not have children. Your definition should be recursive, based on the recursive definition of the set F.

(c) Define the function $numinternal(t)$, for $t \in F$, giving the number of "internal" nodes of the full binary tree t; that is, $numinternal(t)$, for $t \in F$, gives the number of nodes in t that have children. Your definition should be recursive, based on the recursive definition of the set F.

(d) Prove by structural induction that for any full binary tree t in F, $numleaves(t) = 1 + numinternal(t)$.