Problem Set 2

Due: 2:35pm in lecture, February 15, 2000.

Remember to include a *collaboration statement* on the front page of your solutions. (See Handout 2, *Course Information*.)

Background Reading:

Rosen, *Discrete Mathematics and Its Applications*:
Sections 1.4, 1.5, 1.6

Velleman, *How to Prove It*:
Sections 1.3, 1.4, 2.3

Note: The background reading reviews elementary mathematical topics that should be familiar to you; you can skim these. Do look more closely at any topics that seem unfamiliar, and if there are many such topics, we encourage you to discuss this with your TA so he can help you plan a catch-up strategy.

Optional Problems:

Rosen, *Discrete Mathematics and Its Applications*:
Section 1.4, exercises 5, 7, 9, 13, 17, 23
Section 1.5, exercises 3, 7, 15, 17, 19, 21
Section 1.6, exercises 1, 7, 10, 11, 13, 15, 17

Velleman, *How to Prove It*:
Section 1.4, exercises 1, 3, 8
Section 2.3, exercises 3, 6, 7

Reading:

Rosen, *Discrete Mathematics and Its Applications*:
Induction: Sections 3.2, 3.3
Relations: Section 6.1

Velleman, *How to Prove It*:
Induction: Sections 6.1–6.4
Relations: Sections 4.1–4.2

Required Problems:

Problem 1. Suppose you take a piece of paper and draw a bunch of straight lines, each in its own arbitrary direction, that completely cross the paper. This divides the paper up into polygonal regions. Prove by induction that you can always color the various regions using only *two* colors, so that any two regions that share a boundary line are different colors. Note that this need not be true of regions that share only a boundary point.

Problem 2. Prove by induction that $\forall n \geq 1, [n^3 + (n+1)^3 + (n+2)^3$ is evenly divisible by 9].

Problem 3. Prove by induction on n that for all $n \geq 0$, the inequality $|x| + |y| \leq n$ has exactly $2n^2 + 2n + 1$ solutions where x and y are integers.

Problem 4. The 6.042 mascot, Math Moose, made an amazing observation while playing with his collection of unit squares over the weekend.

First, Math Moose put his favorite unit square down on the floor as in diagram (a) of Figure 1. He noticed that the length of the periphery of the resulting shape was 4, an even number. Next, he put a second unit square down next to the first so that the two squares shared an edge as in diagram (b). He noticed that the length of the periphery of the resulting shape was now 6, which is also an even number. (The periphery of each shape in the figure is indicated by a thicker line.) Math Moose continued to place squares so that each new square shared an edge with at least one previously-placed square and no squares overlapped. Eventually, he arrived at the shape in diagram (c). He realized that the length of the periphery of this shape was 36, which is again an even number.

Math Moose spent the rest of the weekend trying to create a shape with an odd-length periphery, without any success. Help him out by proving by induction on the number of squares that the periphery length is always even, no matter how many squares he places or how he arranges them.

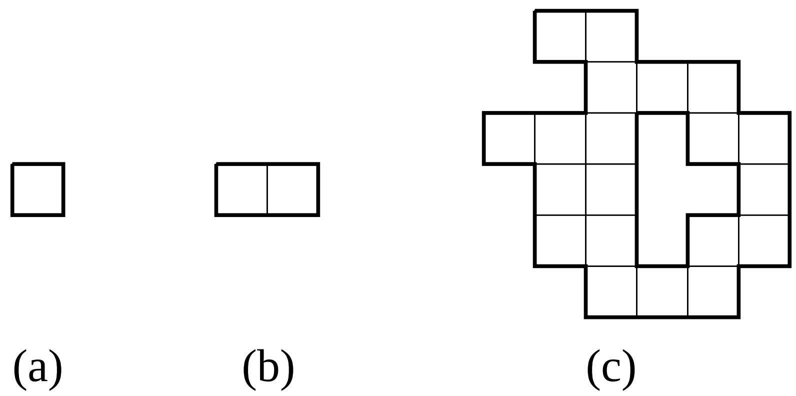
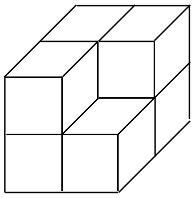


Figure 1: Some shapes that Math Moose created.

Problem 5. The C.I.A. wants you to design plans to build (for an arbitrary $k \geq 1$) a $2^k \times 2^k \times 2^k$ cubic foot granite cube that is solid except for a missing single cubic foot that is one of the eight cubic feet nearest the center of the cube. It will have one of its corners exactly at the center of the large cube. The C.I.A. is not telling you what is to be placed inside the cube.

You are to build this by cementing together *basic blocks* that are 2-foot × 2-foot × 2-foot cubes, with one one-foot cube missing from its corner — see the diagram below.



Describe your method for doing this. Your plans should be described inductively on k.

Problem 6. One plan considered by the 6.042 staff, then dropped, involved requiring students in tutorials to work in groups of either exactly 4 or exactly 7 students. We abandoned this plan when we realized that a tutorial section of size 13 could not be split up in this way.

(a) What is the smallest number k such that all tutorials of k or more students *can* be split into teams of 4 and 7?

(b) Prove that k is the smallest such number.

(c) Prove (by induction) that k does work, that is, that all tutorials of k or more students can be split into teams of size 4 and 7.

Problem 7. Let F_n be the n^{th} Fibonacci number. Recall that the Fibonacci numbers are defined recursively as follows: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, for all $n \in N, n \geq 2$.

(a) Prove that for all $n \in N$, $\sum_{i=0}^{n} F_i = F_{n+2} - 1$.

(b) Find a formula for $\sum_{i=0}^{n} F_{2i}$ and prove that the formula is correct.

Problem 8. A sequence $G_0, G_1, G_2, \ldots$ is defined recursively as follows: $G_0 = 0$, $G_1 = 1$, and $G_n = 5G_{n-1} - 6G_{n-2}$, for every $n \in N, n \geq 2$.

Prove that for all $n \in N$, $G_n = 3^n - 2^n$.

Problem 9. Construct the following rankings for n-player tournaments, or show that this is not always possible.

(a) Assume that one player, Enrique, wins all his matches, and another, Ricky, wins all but one of his matches. Construct a ranking that is locally fair, and in which Enrique is first and Ricky second.

(b) Assume that there are no sets of 3 players, A, B, and C, such that A beats B, B beats C, and C beats A. That is, $\nexists A, B, C, [AbeatsB \wedge BbeatsC \wedge CbeatsA]$. Construct a ranking that is extremely fair, in the sense that any player X that beats another player Y appears higher than Y in the rankings.

Problem 10. The set W of "well-formed parenthesis strings" can be defined by:

1. empty string is in W, and
2. if s and t are in W then the string $(s)t$ is in W.

Use structural induction to prove that: In any well-formed parenthesis string, any prefix has at least as many left parentheses as right parentheses.

Problem 11. The recursive definition of the set F of full binary trees is like the one for binary trees, except that we only allow the recursive definition step that includes both a left and a right child.

(a) Give the structural inductive definition.

(b) Define the function numleaves F, giving the number of "leaves", that is, the number of nodes that have no children. Your definition should be recursive, based on the recursive definition of F.

(c) Define the function numinternal on F, giving the number of "internal" nodes, that is, the number of nodes that do have children.

(d) Prove by structural induction that for all full binary trees t, $\text{numleaves}(t) = 1 + \text{numinternal}(t)$.