Mini-Quiz 7

1. Write your name:

2. (Rosen, Section 1.8, Problem 13) Show that 2^n is $O(3^n)$ but that 3^n is not $O(2^n)$.

Tutorial 7 Problems

Problem 1 Riemann's Zeta Function $\zeta(k)$ is defined to be the infinite summation:

$$1 + \frac{1}{2^k} + \frac{1}{3^k} \cdots = \sum_{j \geq 1} \frac{1}{j^k}$$

Prove that

$$\sum_{k \geq 2} (\zeta(k) - 1) = 1$$

Hint: Recall from problem set 5, problem 2, that $\sum_{k=2}^{n} \frac{1}{k(k-1)} = 1 - \frac{1}{n}$.

Solution.

$$\begin{aligned}
\sum_{k \geq 2} (\zeta(k) - 1) &= \sum_{k \geq 2} \left[\left(\sum_{j \geq 1} \frac{1}{j^k} \right) - 1 \right] \\
&= \sum_{k \geq 2} \sum_{j \geq 2} \frac{1}{j^k} \\
&= \sum_{j \geq 2} \sum_{k \geq 2} \frac{1}{j^k} \\
&= \sum_{j \geq 2} \frac{1}{j^2} \sum_{k \geq 0} \frac{1}{j^k} \\
&= \sum_{j \geq 2} \frac{1}{j^2} \cdot \frac{1}{1 - 1/j} \\
&= \sum_{j \geq 2} \frac{1}{j(j-1)} \\
&= \lim_{n \to \infty} \sum_{j=2}^{n} \frac{1}{j(j-1)} \\
&= \lim_{n \to \infty} (1 - \frac{1}{n}) \\
&= 1
\end{aligned}$$

$\square$

Problem 2 Linear Recurrences

Math Moose has discovered that the fluctuation in the price of stocks in his portfolio (in particular the stock prices of start-up companies that have recently gone public) satisfy a particular temporal property. The price fluctuation at any given day is twice that of the previous day, plus four times that of the day before that, plus negative eight times the day before that. In order to test out his discovery Math Moose has performed a series of tests of the following nature: he buys one stock of a particular start-up company and evaluates the price fluctuation on each successive day. He has found that the fluctuation of the stock price the day he buys the stock is $f(0) = 0$, the next day it's $f(1) = 1$, and the day after that it's $f(2) = 2$. Given this information can you come up with a closed-form solution for the price fluctuation of start-up stocks as discovered by Math Moose?

(a) Determine the linear recurrence, $f(n)$, corresponding to the fluctuation in the price of start-up company stocks in day n. What are the boundary conditions of $f(n)$ as discovered by Math Moose?

Solution.

The linear recurrence , $f(n)$, corresponding to price fluctuation of a start-up stock in day n is as follows:

$$f(n) = 2f(n-1) + 4f(n-2) - 8f(n-3)$$

with $f(0) = 0$, $f(1) = 1$, and $f(2) = 1$.

(b) Determine the characteristic equation of the linear recurrence $f(n)$ and find it's roots. **Hint:** The characteristic equation has a repeated root.

Solution.

In order to find the characteristic equation, we guess at the solution $f(n) = ca^n$. By plugging in this solution to the linear recurrence above, we obtain the characteristic equation $r^3 - 2r^2 - 4r + 8 = 0$. Since the characteristic equation has a repeated root, by plugging in $(r - r_{1,2})^2(r - r_3) = 0$ and equating coefficients, we obtain the roots $r_{1,2} = 2$ and $r_3 = -2$.

(c) Determine the recurrence solution.

Solution.

The solution is of the general form:

$$f(n) = c_1 2^n + c_2 n 2^n + c_3(-2)^n$$

and the coefficients c_1, c_2, and c_3 are determined from the boundary conditions $f(0) = 0$, $f(1) = 1$, and $f(2) = 2$.

Plugging in for the boundary cases in the general form gives the following system of equations:

$$\begin{aligned} 0 &= c_1 + c_3 \\ 1 &= 2c_1 + 2c_2 - 2c_3 \\ 2 &= 4c_1 + 8c_2 + 4c_3 \end{aligned}$$

The solution to this system of equations is $c_1 = \frac{1}{8}$, $c_2 = \frac{1}{4}$, and $c_3 = -\frac{1}{8}$. Therefore, the complete solution to the recurrence is:

$$f(n) = \frac{1}{8}2^n + \frac{1}{4}n2^n - \frac{1}{8}(-2)^n$$

□

(d) After testing his theory out, Math Moose has discovered that he has to adjust his way of evaluating the fluctuation of the stock price of start-up company founded by MIT graduates. He has noticed that MIT start-ups outperform other start-ups. Thus, he adjusts his scheme so that the price fluctuation for MIT start-up companies is increased by a constant dollar amount equal to 3 dollars each day.

What is the resulting inhomogeneous linear recurrence corresponding to Math Moose's scheme?

Solution.

The resulting recurrence is of the form:

$$f(n) = 2f(n-1) + 4f(n-2) - 8f(n-3) + 3$$

□

(e) Find the particular solution to the resulting inhomogeneous linear recurrence.

Solution.

Since we always choose a particular solution of the form of the extra term of the recurrence, we try the solution $f(n) = c$. Plugging into the resulting recurrence we get the equation:

$$c = 2c + 4c - 8c + 3$$

Solving the equation for c we get $c = 1$. Thus, the particular solution of the inhomogeneous linear recurrence is $f(n) = 1$.

□

(f) What is the complete solution to the resulting inhomogeneous linear recurrence?

Solution.

The solution is of the general form:

$$f(n) = c_1 2^n + c_2 n 2^n + c_3(-2)^n + 1$$

and the coefficients c_1, c_2, and c_3 are determined from the boundary conditions $f(0) = 0$, $f(1) = 1$, and $f(2) = 2$.

Plugging in for the boundary cases in this general form gives the following system of equations:

$$\begin{aligned} 0 &= c_1 + c_3 + 1 \\ 1 &= 2c_1 + 2c_2 - 2c_3 + 1 \\ 2 &= 4c_1 + 8c_2 + 4c_3 + 1 \end{aligned}$$

The solution to this system of equations is $c_1 = -\frac{13}{16}$, $c_2 = \frac{5}{8}$, and $c_3 = -\frac{3}{16}$. Therefore, the complete solution to the recurrence is:

$$f(n) = -\frac{13}{16}2^n + \frac{5}{8}n2^n - \frac{3}{16}(-2)^n + 1$$

□

(g) Do you believe Math Moose's scheme for determining the price fluctuation of start-up stocks?

Solution.

Must determine whether $f(n)$ is monotonically increasing... If it is then that would not seem right because it would mean that the stock price monotonically increases.

□

Problem 3 Pigeonhole Principle

(a) Let A be any set of $n+1$ numbers from the set $\{1, 2, \ldots, 2n\}$. Prove that there are always two numbers in A such that one divides the other.

Solution.

Ref. "Proofs from THE BOOK".

We use the pigeonhole principle as follows: Let any number of A, a, be represented in the form $a = 2^k m$, where m is an odd number between 1 and $2n-1$. Since there are $n+1$ numbers in Ak but only n different odd parts, there must be two numbers in A with the same odd part. Hence, one is a multiple of the other.

$\square$

(b) Let A be any set of n numbers from the set $\{1, 2, \ldots, 2n\}$. Prove that there are **not** always two numbers in A such that one divides the other.

Solution.

Ref. "Proofs from THE BOOK".

Consider the set $A = \{n+1, n+2, \ldots, 2n\}$. In this case, the smallest number $n+1$ could not possibly divide the others because it can not divide even the largest number $2n$ (unless $2n = n+1$, which is only true for $n = 1$), since $2(n+1) > 2n$. It follows, that none of the numbers can divide any of the other numbers.

$\square$