Problem Set 3 Solutions

Required Problems:

Problem 1 Prove that for any sets A, B, C, and D, if $A \times B$ and $C \times D$ are disjoint, then either A and C are disjoint or B and D are disjoint.

Solution.
We will prove the contrapositive. In other words,
prove: $[(A \cap C) \neq \emptyset \wedge (B \cap D) \neq \emptyset] \implies [(A \times B) \cap (C \times D) \neq \emptyset]$
assume: $[(A \cap C) \neq \emptyset \wedge (B \cap D) \neq \emptyset]$
$\therefore \exists a(a \in A \wedge a \in C) \wedge \exists b(b \in B \wedge b \in D)$.
then, $(a, b) \in A \times B$ by definition of Cartesian product, Velleman 158.
and, $(a, b) \in C \times D$ by definition of Cartesian product, Velleman 158.
$\therefore (a, b) \in [(A \times B) \cap (C \times D)]$ so $[(A \times B) \cap (C \times D) \neq \emptyset]$
QED

□

Problem 2 Find the flaws in the following fallacious proofs of putative theorems. For each, if the theorem is false, give a counterexample; if it is correct, give a correct proof.

(a) *Theorem?* For any sets A, B, C, and D, $(A \cup C) \times (B \cup D) \subseteq (A \times B) \cup (C \times D)$?

Proof. Suppose $(x, y) \in (A \cup C) \times (B \cup D)$. Then $x \in A \cup C$ and $y \in B \cup D$, so either $x \in A$ or $x \in C$, and either $y \in B$ or $y \in D$. We consider these cases separately.

Case 1. $x \in A$ and $y \in B$. Then $(x, y) \in A \times B$.
Case 2. $x \in C$ and $y \in D$. Then $(x, y) \in C \times D$.

Thus, either $(x, y) \in A \times B$ or $(x, y) \in C \times D$, so $(x, y) \in (A \times B) \cup (C \times D)$. ■

Solution.
The theorem does not hold. The cases do not exhaust all of the possibilities. The correct way to write the proof would be to include first the possibilities that A, B, C, and D are null and then trying out all *four* cases, $x \in A \wedge y \in B$, $x \in A \wedge y \in D$, $x \in C \wedge y \in B$, and $x \in C \wedge y \in D$.
Consider the case where $A = \emptyset$, $C = \{1\}$, $B = \{2\}$, and $D = \{4\}$. Then, the LHS (left-hand-side) gives $\{(1, 2), (1, 4)\}$, while the RHS gives $\{(1, 4)\}$. The statement of the theorem clearly does not hold in this case.

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(b) *Theorem?* Suppose R is a relation on A. If R is symmetric and transitive, then R is reflexive.

Proof. Let x be an arbitrary element of A. Let y be any element of A such that xRy. Since R is symmetric, it follows that yRx. Then by transitivity, since xRy and yRx. We conclude that xRx. Since x was arbitrary, we have shown that $\forall x \in A(xRx)$, so R is reflexive. ■

Solution.
The theorem does not hold. Suppose $A = \{a, b, c, d\}$ and we are given the relation $R = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$. Then R is symmetric ["$\forall x \in A \forall y \in A(xRy \implies yRx$"] and transitive ["$\forall x \in A \forall y \in A \forall z \in A((xRy \wedge yRz) \implies xRz)$"]. However, it is not reflexive, as it is not the case that "$\forall x \in A(xRx)$", as $(d, d) \notin R$.

□

(c) *Theorem?* Suppose R is a reflexive, antisymmetric relation on A. Let S be the transitive closure of R. Then S is a partial order on A.

Proof. R is already reflexive and antisymmetric. To form the relation S we add more ordered pairs to make it transitive as well. Thus, S is reflexive, antisymmetric, and transitive, so it is a partial order. ■

Solution.
The theorem does not hold. Suppose we have the relation $R = \{(a, a), (b, b), (c, c), (a, b), (b, c), (c, a)\}$. This relation is both reflexive and antisymmetric on $A = \{a, b, c\}$. However, taking its transitive closure would break the antisymmetry. For example, the transitive closure T of R includes (a, c), since there is a walk from a to c in R. Since T also contains (c, a), it is not antisymmetric.

□

Problem 3 Suppose R and S are binary relations on a set A. For each part, give either proofs or counterexamples to justify your answers.

(a) Suppose R and S are reflexive. Must $R \cap S$ be reflexive? What about $R \cup S$? What about R^{-1}? What about $R \circ S$?

Solution.
$R \cap S$: **Yes.** $\forall a \in R\ (a, a) \wedge \forall a \in S\ (a, a)$. Therefore, $R \cap S$ contains (a, a) for every $a \in A$.
$R \cup S$: **Yes.** Their intersection contained $(a, a) \forall a \in A$, so their union does too.
R^{-1}: **Yes.** For every $(a, a) \in R$, it holds that $(a, a) \in R^{-1}$ by the definition of inverse. Thus, since $\forall a \in A(a, a) \in R$, it holds that $\forall a \in A(a, a) \in R^{-1}$, so the inverse is reflexive.
$R \circ S$: **Yes.** It holds that $\forall a \in A((a, a) \in S \wedge (a, a) \in R)$. Thus, $\forall a \in A\ \exists b \in A\ ((a, b) \in S \wedge (b, a) \in R)$, so that $\forall a \in A\ ((a, a) \in R \circ S)$ by the definition of composition.

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(b) Suppose R and S are symmetric. Must $R \cap S$ be symmetric? What about $R \cup S$? What about R^{-1}? What about $R \circ S$?

Solution.
$R \cap S$: **Yes.** If $(x, y) \in R \cap S$, we must prove that $(y, x) \in R \cap S$. We have that xRy (in which case yRx as well) and xSy (in which case ySx as well). Since $yRx \wedge ySx$, we have $(y, x) \in R \cap S$.
$R \cup S$: **Yes.** Suppose $(x, y) \in R \cup S$. Therefore, either xRy (in which case yRx and $\therefore (y, x) \in R \cup S$) or xSy (in which case ySx and $\therefore (y, x) \in R \cup S$). So by cases $(y, x) \in R \cup S$.
R^{-1}: **Yes.** Suppose $(x, y) \in R^{-1}$. By the definition of inverse, this means that yRx. But since R is symmetric, this means that xRy. By the definition of inverse again, this means that $(y, x) \in R^{-1}$. So R^{-1} is symmetric.
$R \circ S$: **No.** Suppose $A = \mathbb{N}$, $R = \{(b, c), (c, b)\}$, and $S = \{(a, b), (b, a)\}$. Then $R \circ S = \{(a, c)\}$, which is not symmetric.

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(c) Suppose R and S are transitive. Must $R \cap S$ be transitive? What about $R \cup S$? What about R^{-1}? What about $R \circ S$?

Solution.
$R \cap S$: **Yes.** Suppose there is a walk from x to y in $R \cap S$. We need to prove $(x, y) \in R \cap S$. In order for there to be a walk from x to y in $R \cap S$, there must have been a walk from x to y in both of R and S (otherwise, those walks would not appear in their intersection as walks). Thus, xRy and xSy, so $(x, y) \in R \cap S$ as needed.
$R \cup S$: **No.** Suppose $R = \{(a, b), (b, c), (a, c)\}$ and $S = \{(b, c), (c, a), (b, a)\}$. Then $U = R \cup S = \{(a, b), (b, a), (b, c), (c, a), (a, c)\}$, so that cUa and aUb but $\neg(cUb)$.
R^{-1}: **Yes.** Suppose $(a, b) \in R^{-1} \wedge (b, c) \in R^{-1}$. We must prove that $(a, c) \in R^{-1}$. We know from the definition of inverse that $bRa \wedge cRb$. Furthermore, since R is transitive, we have that cRa. By the definition of inverse again we have that $(a, c) \in R^{-1}$, as needed.
$R \circ S$: **No.** Suppose $R = \{(b, a), (b, c), (c, a)\}$ and $S = \{(a, b), (a, c), (b, c)\}$. Then $(b, a) \in R \circ S$ and $(a, c) \in R \circ S$ but $(b, c) \notin R \circ S$. Transitivity fails for the composition.

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(d) Suppose R and S are equivalence relations. Must $R \cap S$ be an equivalence relation on A? What about $R \cup S$? What about $R \circ S$?

Solution.
$R \cap S$: **Yes.** Here we are supposing R and S are reflexive, symmetric, and transitive. Since we showed above that these properties hold over intersection, $R \cap S$ is reflexive, symmetric, transitive, and therefore an equivalence relation as well.
$R \cup S$: **No.** Suppose $R = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$ and $S = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}$. Then $(c, b) \in R \cup S$ and $(b, a) \in R \cup S$ but $(c, a) \notin R \cup S$. Transitivity fails and $R \cup S$ is not an equivalence relation.
$R \circ S$: **No.** Suppose R and S are as in the above counterexample. Then $(a, b) \in R \circ S$ and $(b, c) \in R \circ S$ but $(a, c) \notin R \circ S$. Transitivity fails and $R \circ S$ is not an equivalence relation.

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Problem 4 Suppose R is a partial order on A and S is a partial order on B. Define a relation T on $A \times B$ as follows: $T = \{((a,b),(a',b')) \in (A \times B) \times (A \times B) \mid aRa' \text{ and } bSb'\}$. Show that T is a partial order on $A \times B$. If both R and S are total orders, will T also be a total order?

Solution.
To show T is a poset on $A \times B$, we must show it is reflexive, symmetric, and transitive on $A \times B$.
Reflexive: Choose an arbitrary element of $A \times B$, if one exists (if none exists, there's nothing to prove and T is reflexive). Call this element (x,y). To prove T is reflexive, we must prove $((x,y)T(x,y))$. Given we have (x,y) chosen, we know xRx and ySy as both R and S are reflexive. Thus, by the definition of T, $((x,y),(x,y)) \in T$ as it is an element of $(A \times B) \times (A \times B)$ such that $xRx \wedge ySy$. Therefore, T is reflexive.
Antisymmetric: We must show that $\forall (a,b) \in (A \times B)\ \forall (c,d) \in (A \times B)\ (((a,b)T(c,d) \wedge (c,d)T(a,b)) \implies (a,b) = (c,d))$. Suppose we choose an arbitrary $a \in A$, $b \in B$, $c \in A$, and $d \in B$ such that the above premise holds. If no such elements exist, T is trivially antisymmetric. We have that $aRc \wedge cRa$ from the definition of T, and thus $a = c$ as R is antisymmetric. Furthermore, $b = d$ by similar rationale. Thus, $(a,b) = (c,d)$, as needed. T is antisymmetric.
Transitive: We must show that $\forall (a,b) \in (A \times B)\ \forall (c,d) \in (A \times B)\ \forall (e,f) \in (A \times B)\ (((a,b)T(c,d) \wedge (c,d)T(e,f)) \implies (a,b)T(e,f))$. Let's arbitrarily choose a particular $a \in A$, $b \in B$, $c \in A$, $d \in B$, $e \in A$, and $f \in B$ such that the premise of the above implication holds. If there are no such elements, there is nothing to prove and T is transitive. We know that since aRc and cRe (by the definition of T), aRe (by the definition of R being transitive). Similarly, bSf as S is transitive. But since $aRe \wedge bSf$, $((a,b),(e,f)) \in T$ by the definition of T. Therefore, T is transitive.
No, T is not a total order. Suppose you have $R = \{(a,a),(a,b),(b,b)\}$ (lexicographically less than or equal to) and $S = \{(1,1),(1,2),(2,2)\}$ (numerically less than or equal to). Then if we form T as defined, there is no way to compare the points $(b,1)$ and $(a,2)$.

□

Problem 5 Suppose R is a relation on A, and let S be the transitive closure of R. Prove that if R is symmetric, then so is S. (Hint: Assume that R is symmetric. Prove that $R \subseteq S^{-1}$ and S^{-1} is transitive. What can you conclude about S and S^{-1}?)

Solution.

Proof. Assume that R is symmetric. We can prove that $R \subseteq S^{-1}$. Suppose arbitrary $(a,b) \in R$. Then $(b,a) \in R$ as well. So, both (a,b) and (b,a) are in S as $R \subseteq S$ by definition of closure. Likewise, (b,a) and (a,b) are also in S^{-1} by the defintion of inverse. Thus, $R \subseteq S^{-1}$.
We must now prove that S^{-1} is transitive. Suppose we choose an arbitrary three elements $a \in A$, $b \in A$, and $c \in A$ such that $((a,b) \in S^{-1} \wedge (b,c) \in S^{-1})$. This means that $(b,a) \in S \wedge (c,b) \in S$ by definition of inverse. But since S is transitive this means $(c,a) \in S$. Thus, by definition of inverse $(a,c) \in S^{-1}$. This is the necessary condition to satisfy transitivity. Thus, S^{-1} is transitive.
We know that the cardinality of S and its inverse are both the same. Since S was the transitive closure of R, it is the smallest set such that $R \subseteq S$ and S is transitive. Since S^{-1} has the same cardinality as S, $R \subseteq S^{-1}$, and S^{-1} is a transitive superset of (or equal to) R; S^{-1} is also a transitive closure of R. But the transitive closure of R is unique. So, $S = S^{-1}$.
Therefore, since S is the same set as its inverse, S is symmetric. *QED*. ■

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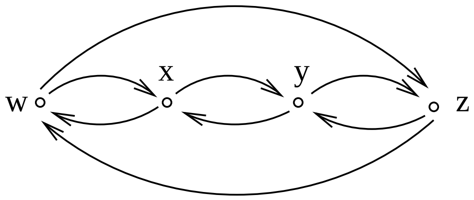
Problem 6 A relation R on a set A is *weakly transitive* if for all $a, b, c, d \in A$, the relations aRb, bRc, and cRd imply the relation aRd.

Prove or disprove the following statements.

(a) Every symmetric, weakly transitive relation is transitive.

Solution. The claim is false. The diagram below shows a counterexample.



(b) Every reflexive, symmetric, weakly transitive relation is an equivalence relation.

Solution.

Proof. The relation R is an equivalence relation if it is reflexive, symmetric, and transitive. The first two properties hold by assumption; all that remains is to show that R is a transitive relation. Suppose that the relations aRb and bRc hold. Because R is reflexive, the relation aRa also holds. Because R is weakly transitive, the relations aRa, aRb, and bRc imply the relation aRc. Therefore, R is transitive. ■

Problem 7 Let G and H be undirected graphs with no self-loops. The *product graph* $P = G \times H$ is defined as follows. The vertex set of P is the Cartesian product of the vertex set of G and the vertex set of H. Thus, each vertex of P is an ordered pair (v, w) where v is a vertex of G and w is a vertex of H. The edges in the product graph P are of two types. First, for every edge (v_1, v_2) in G and vertex w in H, there is an edge between (v_1, w) and (v_2, w) in P. Second, for every vertex v in G and edge (w_1, w_2) in H, there is an edge between (v, w_1) and (v, w_2) in P. Note that these two sets of edges are disjoint.

(a) Draw the product of the two graphs shown below. Your diagram must be tidy to receive full credit. In particular, label all the vertices, e.g., $(b, 3)$, and use two different colors or line styles (e.g., dashed and solid) to distinguish the two types of edge defined above.

Solution.

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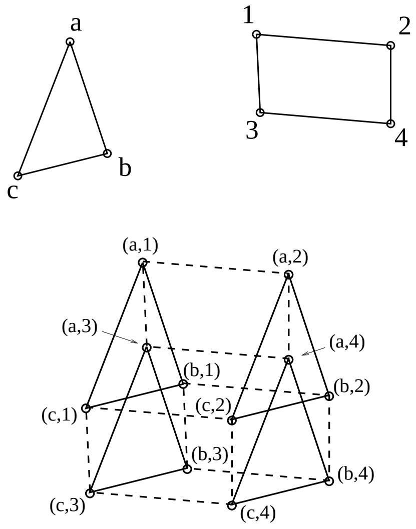

(b) Let G be the graph consisting of two vertices connected by an edge. Let G^n be the product of n copies of G. Use induction to prove that G^n has 2^n vertices and $n2^{n-1}$ edges. (You may assume without proof that the $\times$ operation is associative.)

Solution.

Proof. The proof is by induction on n. Let $P(n)$ be the proposition that G^n has 2^n vertices and $n2^{n-1}$ edges. In the base case, $P(1)$ is true because G^1 has $2^1 = 2$ vertices and $1 \cdot 2^{1-1} = 1$ edge.

In the inductive step, for $n \geq 1$ assume that G^n has 2^n vertices and $n2^{n-1}$ edges to prove that G^{n+1} has 2^{n+1} vertices and $(n+1)2^n$ edges. The remainder of the argument uses the fact that $G^{n+1} = G^n \times G$.

The vertex set of G^{n+1} is the Cartesian product of the vertex set of G^n and the vertex set of G. There are 2^n vertices in G^n by induction, and there are 2 vertices in G by the definition of the graph. Therefore, there are $2^n \cdot 2 = 2^{n+1}$ vertices in G^{n+1}.

The edges of $G^n \times G$ are of two types. First, for each edge (v_1, v_2) in G^n and vertex w in G, there is an edge between (v_1, w) and (v_2, w) in G^{n+1}. Thus, the number of edges of the first type is equal to the number of edges in G^n multiplied by the number of vertices in G, which gives $n2^{n-1} \cdot 2 = n2^n$.

Second, for each vertex v in G^n and edge (w_1, w_2) in G, there is an edge between (v, w_1) and (v, w_2) in G^{n+1}. Thus, the number of edges of the second type is equal to the number of vertices in G^n multiplied by the number of edges in G, which gives $2^n \cdot 1 = 2^n$. Since these two sets of edges are disjoint, the total number of edges in G^{n+1} is equal to $n2^n + 2^n = (n+1)2^n$.

This shows that $P(n)$ implies $P(n+1)$, and the claim is proved by induction. ■

Problem 8 Read the recursive definition of strings over an alphabet Σ in Rosen 3.3, example 10. Define the *reversal* of a string w, written w^R, as follows:

- $\lambda^R = \lambda$
- if $w = vx$ where $v \in \Sigma^*$ and $x \in \Sigma$, then $w^R = xv^R$.

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Prove that $(vw)^R = w^R v^R$

Solution.

Proof.
by **strong induction** on the length, n, of vw.

Base: vw is of length 0. Here, $vw = \lambda$ which means $v = w = \lambda$, so $(vw)^R = (\lambda)^R = \lambda = \lambda\lambda = w^R v^R$.
Inductive step: Assume: $P(n) : \forall i \in \mathbb{N}\ (0 \leq i \leq n) Q(i)$ holds (Inductive Hypothesis).
That is, $Q(i)$: "$(vw)^R = w^R v^R$ *for an arbitrary string vw of length i*"
We now **prove** $P(n+1)$ holds.
Let vw be an arbitrary string of length $n+1$.

Case 1: w is of length exactly 0
then $(vw)^R = (v)^R = \lambda v^R = \lambda^R v^R = w^R v^R$.

Case 2: w is of length exactly 1
then $v \in \Sigma^*$ and $w \in \Sigma$ (w is a single symbol)
$(vw)^R = wv^R$ by the definition of *reversal*
then $w = w^R$ as $w^R = (\lambda w)^R = w\lambda^R = w$ by definition of *reversal* for a single symbol
so $(vw)^R = w^R v^R$

Case 3: w is of length > 1 and $\leq n$
then $v \in \Sigma^*$ and has length ≥ 1
$x \in \Sigma^*$ and $y \in \Sigma$ for some x, y, s.t. $w = xy$
thus, $vw = vxy$ for some $y \in \Sigma$
let $t = vx$ then, $t \in \Sigma^*$ and is of length n by the definition of *length*
then, $vw = ty$ so that $(vw)^R = (ty)^R = yt^R$ by the definition of *reversal*
we know that $t^R = (vx)^R = x^R v^R$ by the Inductive Hypothesis at n.
so, $(vw)^R = yx^R v^R = y^R x^R v^R$ by substitution and $y = y^R$ for a single symbol.
we have that $y^R x^R = (xy)^R = w^R$ by the Inductive Hypothesis at some $i \leq n$
thus, $(vw)^R = w^R v^R$.

Case 4: w is of length $n+1$
we have in this case that $v = \lambda$
thus, $(vw)^R = (\lambda w)^R = (w)^R = w^R \lambda = w^R \lambda^R = w^R v^R$

QED

■

Problem 9 A pair 6.042 students, Nancy and David, have decided to devote some of their spare time this term to establishing dominion over the entire galaxy. Recognizing this as an ambitious project, they worked out the following table of tasks on the back of David's copy of the notes for tutorial 2.

1. **Devise a logo** and cool imperial theme music - 8 days.

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2. **Build a fleet** of Hyperwarp Stardestroyers out of eating paraphernalia swiped from Lobdell - 18 days.

3. **Seize control** of the United Nations - 9 days, after task #1.

4. **Get shots** for Nancy's cat, Muffy - 11 days, after task #1.

5. **Open a Starbucks chain** for the army to get their caffeine - 10 days, after task #3

6. **Train an army** of elite interstellar warriors by dragging people to see *The Phantom Menace* dozens of times - 4 days, after tasks #3, #4, and #5.

7. **Launch the fleet** of Stardestroyers, crush all sentient alien species, and establish a Galactic Empire - 6 days, after tasks #2 and #6.

8. **Defeat Microsoft** - 8 days, after tasks #2 and #6.

Nancy and David want to complete all these tasks in the shortest possible time. However, they have agreed on some constraining work rules.

- Only one person can be assigned to a particular task; they can not work together on a single task.
- Once a person is assigned to a task, that person must work exclusively on the assignment until it is completed. So, for example, Nancy cannot work on building a fleet for a few days, run get shots for Muffy, and then return to building the fleet.

For each task, the table above lists the time required for completion and which other tasks must be finished before the given task is begun.

(a) Express the information in the task list using some type of graph (label the vertices to reflect task lengths).

Solution. The information in the table is represented in the Figure 1 in the form of a directed acyclic graph. Each vertex represents a task, and the weight of a vertex is the completion time. Each directed edge represents a dependency between tasks.

(b) Give some valid order in which the tasks might be completed.

(c) Nancy and David want to know how long conquering the galaxy will take. David suggests dividing the total number of days of work by the number of workers, which is two. What lower bound on the time to conquer the galaxy does this give, and why might the actual time required be greater?

$$\frac{8+18+9+11+10+4+6+8}{2} = 37 \text{ days}$$

If working together and interrupting work on a task were permitted, then this answer would be correct. However, the rules may prevent Nancy and David from both working all the time.

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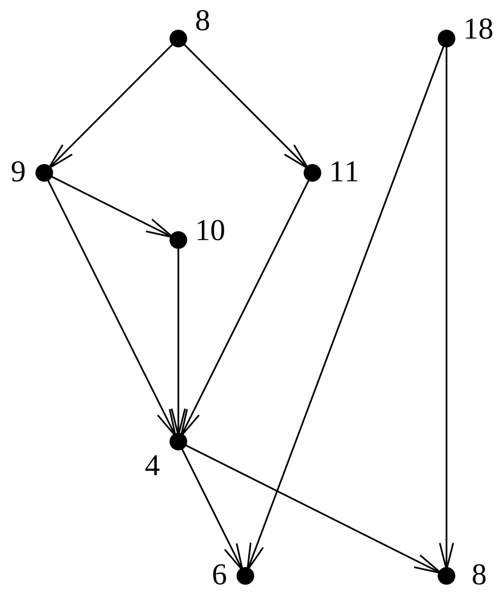

Figure 1: Graph representing the task precedence constraints.

(d) Nancy proposes a different method for determining the duration of their project. She suggests looking at the duration of the "critical path", the longest sequence of tasks such that each depends on the one before. What lower bound does this give, and why might it also be too low?

The longest sequence of tasks is devising a logo (8 days), seizing the U. N. (9 days), opening a Starbucks (10 days), training the army (4 days), and then defeating Microsoft (8 days). Since these tasks must be done sequentially, galactic conquest will require at least 39 days.

If there were enough workers, this answer would be correct; however, with only two workers, Nancy and David may be unable to make progress on the critical path every day.

(e) What is the minimum number of days that Nancy and David need to conquer the galaxy? No proof is required.

40 days. Tasks could be divided as follows:
David: #1 (days 1-8), #3 (days 9-17), #4 (days 18-28), #8 (days 33-40).
Nancy: #2 (days 1-18), #5 (days 19-28), #6 (days 29-32), #7 (days 33-38).

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