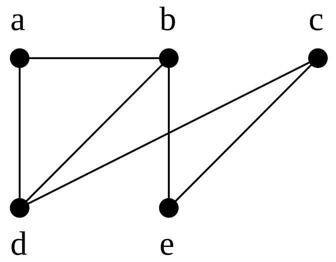
Mini-Quiz 4

1. Write your name:

2. Construct the adjacency matrix that represents the following graph.



Massachusetts Institute of Technology — Handout 12
6.042J/18.062J: Mathematics for Computer Science — February 25, 2000
Professors David Karger and Nancy Lynch

Tutorial 4 Problems

Problem 1 Let L_n be the $n+1$-vertex simple "line" graph consisting of a single simple path of length n. For example, L_4 is shown in the figure below:

The line graph L_4

Let's say a simple graph has "two ends" if it has exactly two vertices of degree one, and all its other vertices have degree two. In particular, for $n \geq 1$, the graph L_n has two ends.

(a) Draw a diagram of the smallest simple graph with two ends which is not isomorphic to any line graph. *Hint*: See part (d).

(b) Explain briefly, but clearly, what is wrong with the following argument:

False theorem: Every simple graph with two ends is isomorphic to L_n for some $n \geq 1$.

False proof: We prove by induction on the number, $n \geq 1$, of edges in a simple graph, that every two-ended graph with n edges is isomorphic to L_n.

(Base case $n = 1$): A simple graph with one edge can only consist of the two vertices connected by that edge and some number of vertices not attached to any edge, i.e., vertices of degree zero. Since a two-ended graph cannot have vertices of degree zero, the only two-ended graph with one edge must consist solely of the two vertices connected by the edge, which makes it isomorphic to L_1.

(Induction case): Assume that $n \geq 1$ and every two-ended graph with n edges is isomorphic to L_n. Now let G_n be any two-ended graph with $n \geq 1$ edges. By hypothesis, G_n is isomorphic to L_n. Suppose an edge is added to G_n to form a two-ended graph G_{n+1}.

Since G_n is isomorphic to L_n, it consists of a simple path of length n. The only way to add an edge to the path and preserve two-endedness is to have that edge go from one end—that is, one of the degree-one vertices—to a new vertex, lengthening the path by one. That is, the resulting $n+1$-edge graph must be a simple path of length $n+1$, so it is isomorphic to L_{n+1}. Q.E.D.

(c) Here is another argument for the same False Theorem as in part (b). Explain exactly where it goes wrong.

False proof 2: Same induction hypothesis and base case as in part (b).

(Induction case): For any $n \geq 1$, let G_{n+1} be any two-ended graph with $n+1$ edges. Let G_n be the graph which results from removing one of the degree-one vertices v of G_{n+1} and the edge $\{v, w\}$ attached to it. So G_n no longer has the vertex, v, of degree one. But the degree of w is one less in G_n than it was in G_{n+1}, so G_n still has two vertices of degree one and one fewer vertex of degree two. Therefore G_n is also two-ended. By induction G_n consists of a simple path of length n. But G_{n+1} is obtained by attaching an edge from one end of the path to a vertex v not on the path, thereby lengthening the path by one. So G_{n+1} is isomorphic to a simple path of length $n+1$; that is, it is isomorphic to L_{n+1}. Q.E.D.

(d) Describe how to make a small revision to one of the false proofs above so that it becomes a correct proof of the theorem "Every *connected* simple graph with two ends is isomorphic to L_n for some $n \geq 1$."

Problem 2 Over the weekend, your 6.042 TAs came across the following puzzle: a 3×3 grid of squares, where each square contains one of the digits in the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, each digit appearing exactly once in the puzzle. Each row can be rotated right or left and each column can be rotated up or down. The effects of such rotations are demonstrated in the following figure.

$$\begin{array}{|c|c|c|}\hline 2 & 3 & 6 \\ \hline 7 & 8 & 9 \\ \hline 5 & 4 & 1 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|}\hline 5 & 3 & 6 \\ \hline 2 & 8 & 9 \\ \hline 7 & 4 & 1 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|}\hline 5 & 3 & 6 \\ \hline 8 & 9 & 2 \\ \hline 7 & 4 & 1 \\ \hline \end{array}$$

Figure 1: Examples of puzzle row and column rotations.

The first picture shows the initial configuration of the puzzle. The leftmost column is then rotated downward to give the middle picture. The 2 and 7 move down and the 5 rotates back up to the top. Then, the middle row is rotated leftward to produce the final picture. The 8 and 9 move left and the 2 rotates back around to the right.

Since discovering the puzzle, your 6.042 TAs have spent hours trying to obtain the following configuration, starting of course from the initial configuration of Figure 1 and using legal row and column rotations.

1	2	3
4	5	6
7	8	9

Sad to say, your 6.042 TAs have failed miserably.

Based on prior experience with puzzles, the 6.042 students believe that the puzzle might be designed in a way that makes this configuration impossible, no matter what sequence of row and column rotations one attempts.

(a) Model the puzzle as a state machine.

(b) Given the initial configuration of the puzzle indicated above, namely 236 789 541, prove that your 6.042 TAs can never obtain the configuration 123 456 789.