Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
Professors David Karger and Nancy Lynch

Handout 11
February 22, 2000

Problem Set 4

Due: 2:35pm in lecture, February 29, 2000.

Self-Study Reading:

Rosen, *Discrete Mathematics and Its Applications*:
Section 7.3

Note: The self-study reading covers basic mathematical topics that we assume you are familiar with and that will not be explicitly covered in the lectures. We recommend that you skim them over and encourage you to discuss any topics that are unclear to you with your TA during the weekly tutorial.

Self-Study Problems:

Rosen, *Discrete Mathematics and Its Applications*:
Section 7.3, exercises 3, 7, 11, 26, 35, 43

Note: The self-study problems are intended for you to get some practice on the mathematical topics covered by the self-study reading. Again, make sure that you understand the concepts presented in the problems and that you discuss any topics that are unclear to you with your TA during the weekly tutorial.

Reading:

Rosen, *Discrete Mathematics and Its Applications*:
Graphs: Sections 7.1, 7.2, 7.4, 7.5, 7.8; Trees: Section 8.1; Computational Processes: Rosen 2.1;

6.042 Fall 97 Lecture notes (available from course page):
Graphs: Lecture 4 handout; Trees and Dags: Lecture 5 handout;

Required Problems:

Remember to include a *collaboration statement* on the front page of your solutions.
(See Handout 1, *Course Information.*)

Problem 1 An edge of a connected graph is called a *cut-edge* if removing the edge disconnects the graph. Prove that an edge e in a connected graph G is a cut-edge if and only if no simple cycle contains e.

Problem 2 Let $G = (V, E)$ be an undirected graph, and let $r : V \to \mathbb{R}$ be a function associating a real number with each vertex. Suppose r satisfies the property that for each $v \in V$ such that v has at least one neighbor, the value $r(v)$ is the average of the values of v's neighbors. Prove that if the graph is connected, then $r(u) = r(v)$ for all $u, v \in V$.

Problem 3 An n-node graph is said to be *tangled* if there is an edge leaving every set of $\lceil \frac{n}{3} \rceil$ or fewer vertices, that is, for every set of at most $\lceil \frac{n}{3} \rceil$ vertices, there is an edge between a vertex in the set and a vertex not in the set. As a special case, the graph consisting of a single node is considered tangled. (Recall that the notation $\lceil x \rceil$ refers to the smallest integer greater than or equal to x.)

(a) Find the error in the proof of the following claim.

Theorem? Every non-empty, tangled graph is connected.

Proof. The proof is by strong induction on the number of vertices in the graph. Let $P(n)$ be the propostion that if an n-node graph is tangled, then it is connected. In the base case, $P(1)$ is true because the graph consisting of a single node is defined to be tangled and is trivially connected.

In the inductive step, for $n \geq 1$ assume $P(1), \ldots, P(n)$ to prove $P(n+1)$. That is, we want to prove that if an $(n+1)$-node graph is tangled, then it is connected. Let G be a tangled, $(n+1)$-node graph. Arbitrarily partition G into two pieces so that the first piece contains exactly $\lceil \frac{n}{3} \rceil$ vertices, and the second piece contains all remaining vertices. Note that since $n \geq 1$, the graph G has a least two vertices, and so both pieces contain at least one vertex. By induction, each of these two pieces is connected. Since the graph G is tangled, there is an edge leaving the first piece, joining it to the second piece. Therefore, the entire graph is connected. This shows that $P(1), \ldots, P(n)$ imply $P(n+1)$, and the claim is proved by strong induction. ■

(b) Draw a tangled graph that is not connected.

(c) An n-node graph is said to be *mangled* if there is an edge leaving every set of $\lceil \frac{n}{2} \rceil$ or fewer vertices. Again, as a special case, the graph consisting of a single node is considered mangled. Prove that every non-empty, mangled graph is connected.

Problem 4 A 2-colorable graph is called *bipartite* (two pieces).

(a) Prove by induction that every tree is bipartite.

(b) Prove that a graph is bipartite if and only if every simple cycle is of even length. (**Hint:** Can v have both an even and an odd length path to w?)

(c) Explain why (a) follows immediately from (b).

Problem 5 The following algorithm, called the *Russian peasants algorithm*, can be used to multiply any two natural numbers x and y using only left bit-shift (i.e., multiply by 2) and right bit-shift (i.e., divide by 2 and drop the remainder, if any) operations. The answer is accumulated in variable a; variables r and s are for temporary storage.

```
$r := x$;
$s := y$;
$a := 0$;
do until $s = 0$:
    if s is even then
        $r := 2r$;
        $s := s/2$;
    else
        $a := a + r$;
        $r := 2r$;
        $s := (s - 1)/2$;
```

The answer xy is the value left in the accumulator a when the procedure terminates.

(a) Model the algorithm as a state machine. That is, define the states Q, the start states Q_0, and the steps δ.

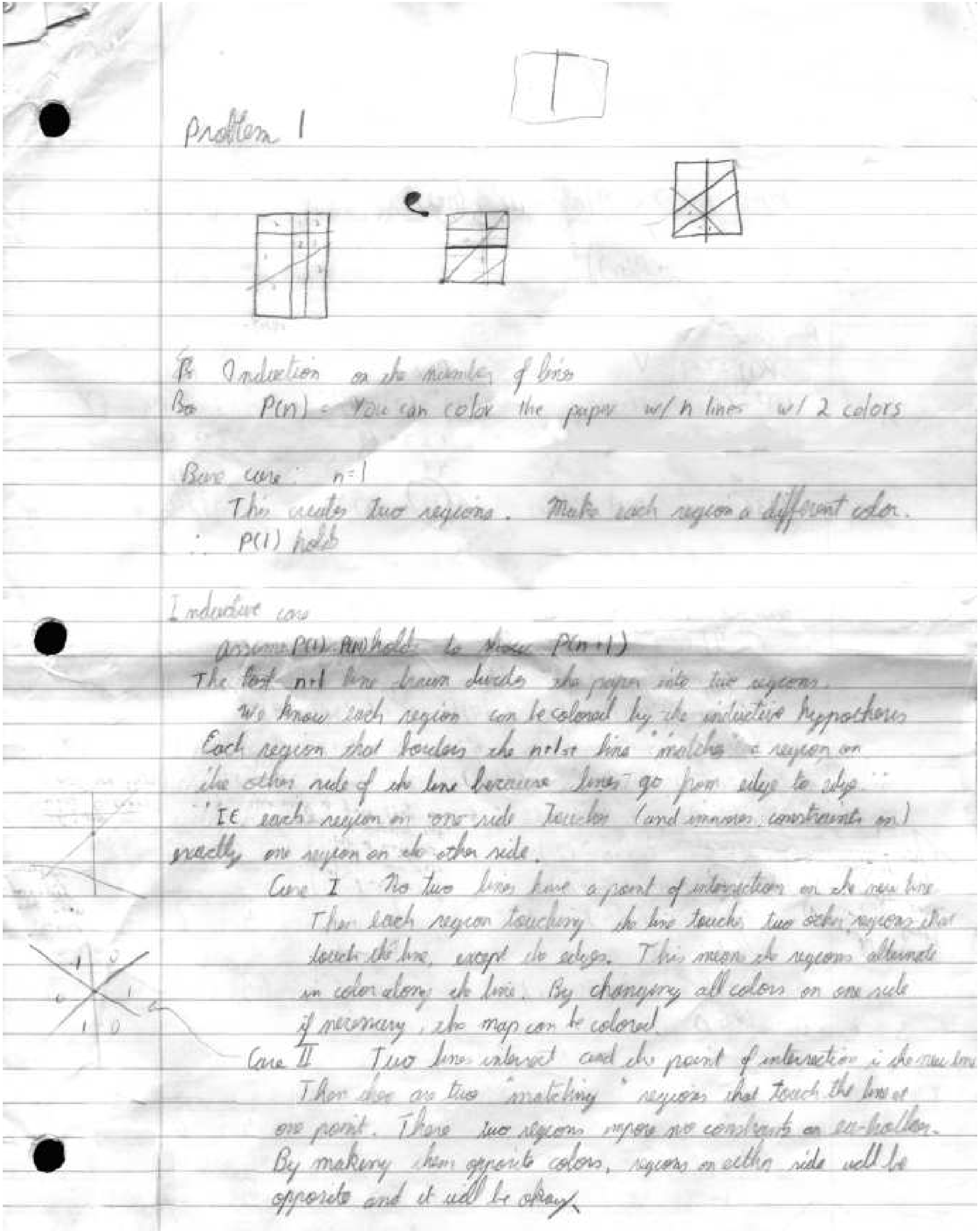
(b) List the sequence of states that appears in an execution of the algorithm for inputs $x = 5$ and $y = 9$.

(c) Prove that the algorithm gives the correct answer if it terminates, that is, that $s = 0$ implies $a = xy$. (Hint: Use induction on the number of steps in the execution. You will have to strengthen the inductive hypothesis.)

(d) Prove that the algorithm terminates.

Problem 6 Attached to this problem set are some students' solutions from a previous problem set. Mark up these solutions, indicating things that are incorrect or unclear, and if possible show how the solution can be made completely correct and clear.

(a) Problem Set 2, Problem 1



Problem 1

Pf Induction on the number of lines

Pf P(n) = You can color the paper w/ n lines w/ 2 colors

Base case: n=1

This creates two regions. Make each region a different color.

∴ P(1) holds

Inductive case

assume P(1)...P(n) holds to show P(n+1)

The first n+1 line drawn divides the paper into two regions.

We know each region can be colored by the inductive hypothesis

Each region that borders the n+1st line "matches" a region on the other side of the line because lines go from edge to edge ∴

IE each region on one side touches (and imposes constraints on) exactly one region on the other side.

Case I No two lines have a point of intersection on the new line

Then each region touching the line touches two other regions that touch the line, except the edges. This means the regions alternate in color along the line. By changing all colors on one side if necessary, the map can be colored.

Case II Two lines intersect and the point of intersection is the new line

Then there are two "matching" regions that touch the line at one point. These two regions impose no constraints on each other. By making them opposite colors, regions on either side will be opposite and it will be okay.

(b) Problem Set 2, Problem 4

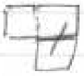
$\boxed{4}$ P(n): For any shape composed of n attached unit squares, the the length of its periphery is even.

$\forall n \in \mathbb{N}\ n > 1,\ P(n)$

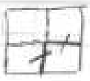
1. Base Case: n=1
   □ has periphery 4, even. ✓

2. Inductive Case:

   1. Case 1: If you add one unit square to another lone unit square (per. len. = 4) you lose 1 len. of per. and gain 3 for a net total 2

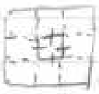


   2. Case 2: Add 1 unit sq. to one sq. of an already existing group lose 1 gain 3 net 2



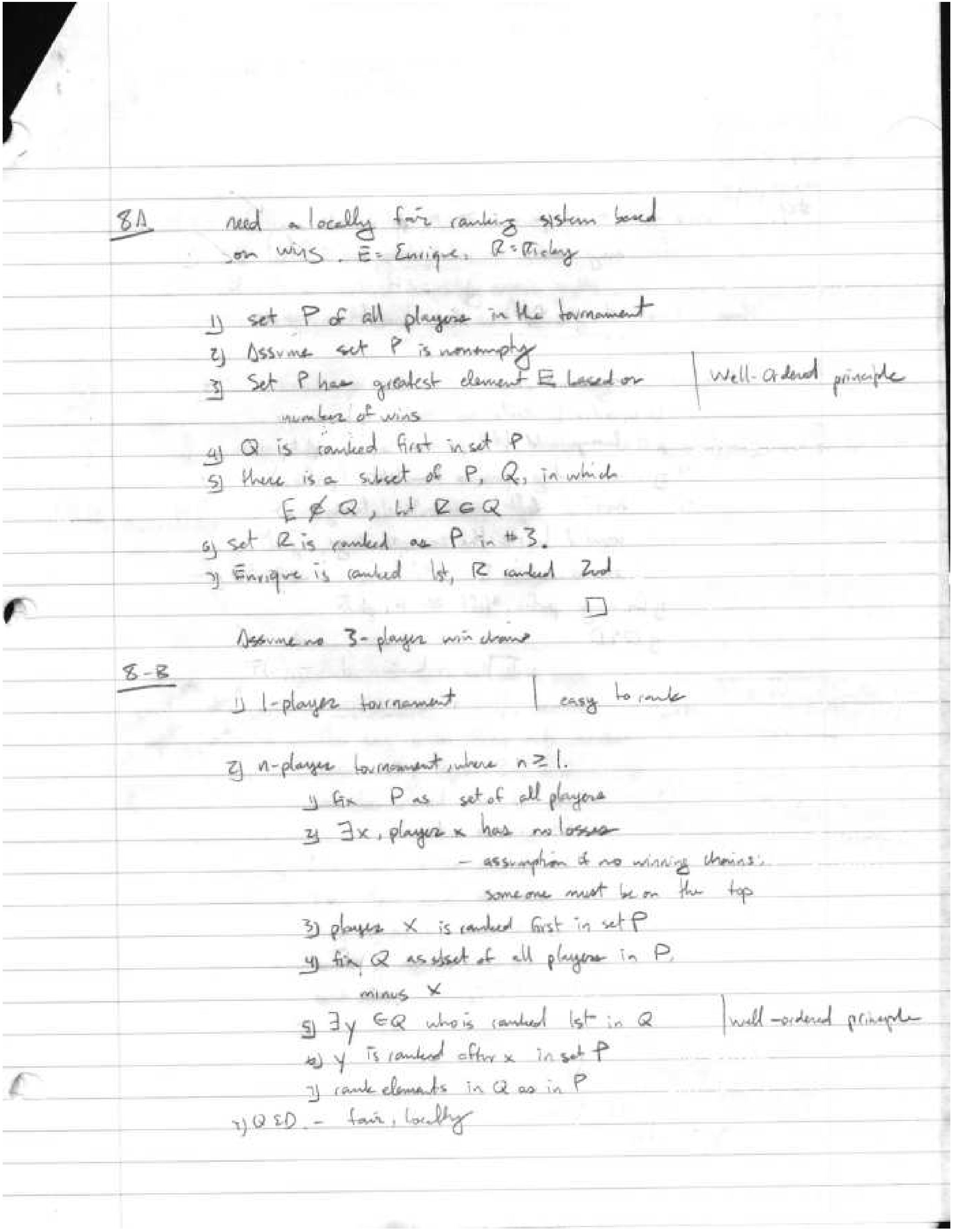
   3. Case 3 Add 1 unit sq. to two sqs of a group - lose 2 gain 2 net 0.

   4. Case 4 Add 1 unit sq. to a group of 5 surrounding sqs, lose 3 gain one net -2



   5. Case 5 Add 1 to a group of 8 surrounding sqs gain 0 lose 4 net -4.
   6. Given a group size n assume it is even in periphery.
   7. A group size (n+1) will always have even periphery, regardless of where on the

(c) Problem Set 2, Problem 8



8A need a locally fair ranking system based on wins. E = Enrique, R = Ricky

1) set P of all players in the tournament
2) Assume set P is nonempty
3) Set P has greatest element E based on number of wins | Well-Ordered principle
4) Q is ranked first in set P
5) there is a subset of P, Q, in which $E \notin Q$, but $R \in Q$
6) set R is ranked as P in #3.
7) Enrique is ranked 1st, R ranked 2nd

$\square$

Assume no 3-player win chains

8-B

1) 1-player tournament | easy to rank

2) n-player tournament, where $n \geq 1$.
   1) fix P as set of all players
   2) $\exists x$, player x has no losses
      - assumption of no winning chains: someone must be on the top
   3) player x is ranked first in set P
   4) fix Q as subset of all players in P, minus x
   5) $\exists y \in Q$ who is ranked 1st in Q | well-ordered principle
   6) y is ranked after x in set P
   7) rank elements in Q as in P

7) QED - fair, locally