Problem Set 4

Due: 2:35pm in lecture, February 29, 2000.

Self-Study Reading:

Rosen, *Discrete Mathematics and Its Applications*:
Section 7.3

Note: The self-study reading covers basic mathematical topics that we assume you are familiar with and that will not be explicitly covered in the lectures. We recommend that you skim them over and encourage you to discuss any topics that are unclear to you with your TA during the weekly tutorial.

Self-Study Problems:

Rosen, *Discrete Mathematics and Its Applications*:
Section 7.3, exercises 3, 7, 11, 26, 35, 43

Note: The self-study problems are intended for you to get some practice on the mathematical topics covered by the self-study reading. Again, make sure that you understand the concepts presented in the problems and that you discuss any topics that are unclear to you with your TA during the weekly tutorial.

Reading:

Rosen, *Discrete Mathematics and Its Applications*:
Graphs: Sections 7.1, 7.2, 7.4, 7.5, 7.8; Trees: Section 8.1; Computational Processes: Rosen 2.1;

6.042 Fall 97 Lecture notes (available from course page):
Graphs: Lecture 4 handout; Trees and Dags: Lecture 5 handout;

Required Problems:

Remember to include a *collaboration statement* on the front page of your solutions. (See Handout 1, *Course Information*.)

Problem 1 An edge of a connected graph is called a *cut-edge* if removing the edge disconnects the graph. Prove that an edge e in a connected graph G is a cut-edge if and only if no simple cycle contains e.

Problem 2 Let $G = (V, E)$ be an undirected graph, and let $r : V \to \mathbb{R}$ be a function associating a real number with each vertex. Suppose r satisfies the property that for each $v \in V$ such that v has at least one neighbor, the value $r(v)$ is the average of the values of v's neighbors. Prove that if the graph is connected, then $r(u) = r(v)$ for all $u, v \in V$.

Problem 3 An n-node graph is said to be *tangled* if there is an edge leaving every set of $\lceil \frac{n}{3} \rceil$ or fewer vertices, that is, for every set of at most $\lceil \frac{n}{3} \rceil$ vertices, there is an edge between a vertex in the set and a vertex not in the set. As a special case, the graph consisting of a single node is considered tangled. (Recall that the notation $\lceil x \rceil$ refers to the smallest integer greater than or equal to x.)

(a) Find the error in the proof of the following claim.

Theorem? Every non-empty, tangled graph is connected.

Proof. The proof is by strong induction on the number of vertices in the graph. Let $P(n)$ be the propostion that if an n-node graph is tangled, then it is connected. In the base case, $P(1)$ is true because the graph consisting of a single node is defined to be tangled and is trivially connected.

In the inductive step, for $n \geq 1$ assume $P(1), \ldots, P(n)$ to prove $P(n+1)$. That is, we want to prove that if an $(n+1)$-node graph is tangled, then it is connected. Let G be a tangled, $(n+1)$-node graph. Arbitrarily partition G into two pieces so that the first piece contains exactly $\lceil \frac{n}{3} \rceil$ vertices, and the second piece contains all remaining vertices. Note that since $n \geq 1$, the graph G has a least two vertices, and so both pieces contain at least one vertex. By induction, each of these two pieces is connected. Since the graph G is tangled, there is an edge leaving the first piece, joining it to the second piece. Therefore, the entire graph is connected. This shows that $P(1), \ldots, P(n)$ imply $P(n+1)$, and the claim is proved by strong induction. ■

(b) Draw a tangled graph that is not connected.

(c) An n-node graph is said to be *mangled* if there is an edge leaving every set of $\lceil \frac{n}{2} \rceil$ or fewer vertices. Again, as a special case, the graph consisting of a single node is considered mangled. Prove that every non-empty, mangled graph is connected.

Problem 4 A 2-colorable graph is called *bipartite* (two pieces).

(a) Prove by induction that every tree is bipartite.

(b) Prove that a graph is bipartite if and only if every simple cycle is of even length. (**Hint:** Can v have both an even and an odd length path to w?)

(c) Explain why (a) follows immediately from (b).

Problem 5 The following algorithm, called the *Russian peasants algorithm*, can be used to multiply any two natural numbers x and y using only left bit-shift (i.e., multiply by 2) and right bit-shift (i.e., divide by 2 and drop the remainder, if any) operations. The answer is accumulated in variable a; variables r and s are for temporary storage.

```
r := x;
s := y;
a := 0;
do until s = 0:
    if s is even then
        r := 2r;
        s := s/2;
    else
        a := a + r;
        r := 2r;
        s := (s − 1)/2;
```

The answer xy is the value left in the accumulator a when the procedure terminates.

(a) Model the algorithm as a state machine. That is, define the states Q, the start states Q_0, and the steps δ.

(b) List the sequence of states that appears in an execution of the algorithm for inputs $x = 5$ and $y = 9$.

(c) Prove that the algorithm gives the correct answer if it terminates, that is, that $s = 0$ implies $a = xy$. (Hint: Use induction on the number of steps in the execution. You will have to strengthen the inductive hypothesis.)

(d) Prove that the algorithm terminates.

Problem 6 Attached to this problem set are some students' solutions from a previous problem set. Mark up these solutions, indicating things that are incorrect or unclear, and if possible show how the solution can be made completely correct and clear.

(a) Problem Set 2, Problem 1

(b) Problem Set 2, Problem 4

(c) Problem Set 2, Problem 8