Mini-Quiz 3

1. Write your name:

2. **Rosen 2.3, Ex. 35:** Show that if $n \mid m$, where n and m are positive integers greater than 1, and if $a = b \pmod{m}$, where a and b are integers, then $a = b \pmod{n}$.

Tutorial 3 Problems

Problem 1 Consider the following putative theorem and the accompanying proof.

Theorem. *For any sets A, B, C, and D, if $A \times B \subseteq C \times D$ then $A \subseteq C$ and $B \subseteq D$.*

Proof. Suppose $A \times B \subseteq C \times D$. Let a be an arbitrary element of A, b be an arbitrary element of B, and $(a, b) \in A \times B$. Since $A \times B \subseteq C \times D$, it follows that $(a, b) \in C \times D$. Therefore, $a \in C$ and $b \in D$. Since a and b were arbitrary elements of A and B, respectively, this shows that $A \subseteq C$ and $B \subseteq D$. ■

Is the proof correct? If so, what proof strategies does it use? If not, can it be fixed? Is the theorem correct?

Problem 2 Relations and Matrices

Let A be the adjacency matrix for a digraph G with n vertices. Note that A is a zero-one valued $n \times n$ matrix. In this problem we use the Boolean "OR" operation, $\vee$, instead of "+" when multiplying zero-one valued matrices (this is called "boolean" matrix product).

(a) For $k \in \mathbb{N}$, let A^k be the kth *Boolean* power of A. Prove that ${A^k}_{ij} = 1$ iff there is a directed path of length k from the ith to the jth vertex of G.

(b) Let $B = A \vee I$ where I is the $n \times n$ identity matrix. Prove that ${B^k}_{ij} = 1$ iff there is a directed path of length $\leq k$ from the ith to the jth vertex of G.

(c) Conclude that ${B^n}_{ij} = 1$ iff there is a directed path from the ith to the jth vertex.

Problem 3 Topological Sort of a DAG

Consider the following definitions of a *cycle* and of a *directed acyclic graph (DAG).*

Definition. *A* ***cycle*** *is a walk that ends where it started (i.e., the last vertex equals the first).*

Definition. *A* ***directed acyclic graph (DAG)*** *is a directed graph with no cycles.*

As an example, Figure 1 contains a picture of a directed acyclic graph.

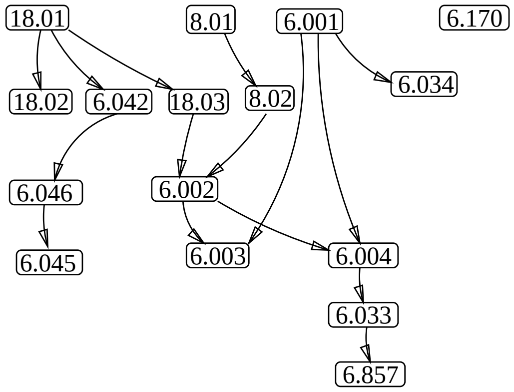

Figure 1: This is a picture of a directed acyclic graph. Vertices are labeled with course numbers. An edge from u to v indicates that course u is a prerequisite for course v.

Now, consider the following definition of a topological sort of the vertices of a DAG G.

Definition. *A* ***topological sort*** *of the vertices of a DAG G is a sequence comprised of all the vertices of G (each appearing once), such that for any two vertices u and v of G, if u appears before v, then there is no edge from v to u.*

We want you to prove the following theorem.

Theorem. *Every DAG has a topological sort.*

The trick that you need to use is that it is possible to pick a "first" vertex for your topological sort and then proceed inductively.

(a) Consider the following definition of a minimal vertex of a DAG. Notice that there can be more than one minimal vertex.

Definition. *A* ***minimal*** *vertex u in a DAG G is a vertex for which there does not exist another vertex v in G such that the edge from v to u is in G.*

What is the minimal vertex of the DAG in Figure 1?

(b) In fact, every DAG has a minimal vertex. Prove the following lemma.

Lemma. *Every DAG has a minimal vertex.*

(c) Give an example of a topological sort of the vertices of the DAG in Figure 1.

(d) Use the above lemma to prove the existence of a topological sort for every DAG.

Problem 4 Scheduling Courses

We consider a simple kind of scheduling problem — for MIT courses with prerequisites. The problem is to determine how a student could complete all of the courses in the fewest terms, under the simplifying, if unlikely, assumption that there is no limit on the number of courses a student can take in a term as long as their prerequisites have all been met. We model this problem with a digraph. An example is shown in Figure 2.

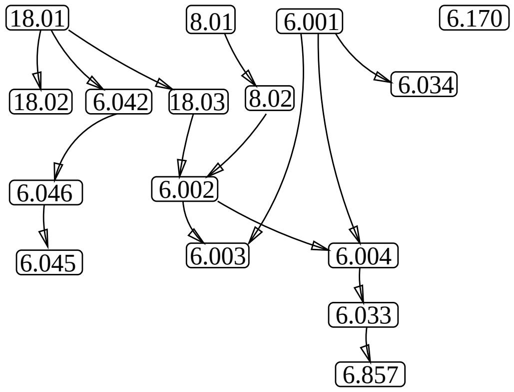

Figure 2: This is a picture of a directed acyclic graph. Vertices are labeled with course numbers. An edge from u to v indicates that course u is a prerequisite for course v.

This example is a DAG. Of course it is possible that the faculty in charge of courses created a cycle of prerequisites; this would mean that it was impossible to fulfill the prerequisites of any of the courses in the cycle before taking the course. Presumably this would not be the intention of the faculty, so we assume that whatever the prerequisite requirements may be, they do not have cycles; so the digraph corresponding to the prerequisites will always be a DAG.

The connectedness relation of a DAG is a *partial order*. The example in Figure 2 suggests the reason for this name. The edges in this graph restrict the order in which a student may take classes. For example, 6.001 must be taken before 6.003. Of course, the student still has some choice about the order; for this reason, the set of classes is said to be "partially" ordered by this system of prerequisites. If we write $u \succeq v$ to mean that $u = v$ or there is a directed path from vertex u to v, then the relation $\succeq$ is obviously *transitive*:

$$[u \succeq v \wedge v \succeq w] \implies u \succeq w,$$

and *antisymmetric*:

$$[u \succeq v \wedge v \succeq u] \implies u = v.$$

Formally, a partial order on a set is just a binary relation which is transitive and antisymmetric.

Two elements u and v are said to be *comparable* under a partial order, $\succeq$, iff $u \succeq v \vee v \succeq u$. When the partial order is the one from the connectedness relation of a DAG, vertices u, v

are comparable precisely when $u = v$ or there is a directed path from u to v or vice-versa. Otherwise, u and v are incomparable. For example, 18.01 and 6.002 are comparable since there is a directed path from 18.01 to 6.002. However, 18.01 and 8.01 are incomparable because there is no directed path from one to the other.

(a) How many terms are necessary to complete all of the courses in Figure 2, assuming prerequisites are observed, but course load is no issue?

(b) Explain why every (finite) DAG has a finite longest walk.

We let t be the number of vertices in the longest walk in the graph. Now at least t terms are necessary to complete all the courses, because at most one course on the chain can be taken each term.

(c) Show that every DAG with a critical path of length t has a schedule with t steps.

(d) Show that there is a schedule for taking courses which achieves the minimum number of terms, t.