Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
Professors David Karger and Nancy Lynch

Handout 8
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# Problem Set 2 Solutions

**test** this is a test

**Problem 1** Suppose you take a piece of paper and draw a bunch of straight lines, each in its own arbitrary direction, that completely cross the paper. This divides the paper up into polygonal regions. Prove by induction that you can always color the various regions using only *two* colors, so that any two regions that share a boundary line are different colors. Note that this need not be true of regions that share only a boundary point.

**Solution.**

We assume that all lines are straight and completely cross the paper. Also, when we say that the regions are colored, we always mean they are colored using at most two colors, *e.g.*, red and green. We say that two regions are *adjacent* if they share a boundary line. Note that two regions cannot share two distinct boundary lines.

Let $P(n)$ be the predicate that the regions formed by any set of $n$ distinct lines can be colored so that (two) adjacent regions are colored with different colors. We prove that $P(n)$ holds for all $n \in \mathbb{N}$ by induction on $n$.

**Base case:** The base case is $P(0)$. There is only one region, and thus there are no adjacent regions. So this case is trivially true. The region may be colored either red or green.

**Inductive step:** We must show that for all $n \in \mathbb{N}$ it is the case that $(P(n) \implies P(n+1))$; that is, if the regions formed by any $n$ distinct lines can be colored so that adjacent regions have different colors, then the regions formed by any $n+1$ distinct lines can be colored so that adjacent regions have different colors.

1. Fix $n \in \mathbb{N}$.
2. Assume that the regions formed by any $n$ distinct lines can be colored so that adjacent regions have different colors.
3. The regions formed by any $n+1$ distinct lines can be colored so that adjacent regions have different colors.
   (a) Fix a set of $n+1$ lines. These lines divide the paper into a set $X$ of regions.
   (b) Select any line $l$ from that set. There are $n$ other lines, which divide the paper into a set $Y$ of regions, such that every region in $X$ is either a region in $Y$ or else it is part of a region in $Y$ that is split by $l$.
   (c) By the inductive hypothesis, the regions in $Y$ can be colored so adjacent regions are different colors. This also gives a coloring for the regions in $X$. Call this coloring of the regions in $X$ the "original coloring."

(d) Line $l$ splits the paper into two sides. Select one of the sides and flip all the colors of the regions on that side. Call this coloring of the regions in $X$ the "new coloring." We prove that, using the new coloring, any adjacent regions in $X$ have different colors.

i. Fix any two adjacent regions in $X$.

**Case 1:** The regions share boundary line $l$. The two regions had the same color in the original coloring. The two regions are on opposite sides of $l$. Therefore, the color in exactly one of the two regions was flipped and they have different colors in the new coloring.

**Case 2:** The regions do not share boundary line $l$. The two regions had different colors in the original coloring, and they are on the same side of $l$. Thus, the color of either none or both regions was flipped and the two regions have different colors in the new coloring.

ii. QED (universal generalization over adjacent regions)

(e) QED (universal generalization over sets of $n + 1$ lines)

4. QED (implication)

QED (induction)

□

**Problem 2** Prove by induction that $\forall\ n \geq 1$, the quantity $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

**Solution.** Let $P(n)$ be the predicate that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9, *i.e.*, $P(n) = 9|(n^3 + (n+1)^3 + (n+2)^3)$. We prove that $P(n)$ holds for all $n \geq 1$ by induction on $n$. In fact, the predicate $P(n)$ is also true for $n = 0$.

**Base Case:** $P(1)$

$P(1) = 9|(1^3 + 2^3 + 3^3) = 9|36 =$ `True`

**Inductive Step:** For all $n \geq 1$, $P(n) \implies P(n+1)$

1. Fix $n \geq 1$.
2. From the inductive hypothesis $P(n)$ holds, *i.e.*, $9|(n^3 + (n+1)^3 + (n+2)^3)$.
3. We must show that $P(n+1)$ holds, *i.e.*, $9|((n+1)^3 + (n+2)^3 + (n+3)^3)$.
   (a) $(n+1)^3 + (n+2)^3 + (n+3)^3 = n^3 + (n+1)^3 + (n+2)^3 + 9(n^2 + 3n + 3)$
   (b) From the inductive hypothesis, $9|(n^3 + (n+1)^3 + (n+2)^3)$.
   (c) Also, it is simple to see that $9|(9(n^2 + 3n + 3))$.
   (d) Since 9 divided both terms, it divides their sum.
   (e) QED
4. QED (implication and UG)

□

**Problem 3** Prove by induction on $n$ that for all $n \geq 0$, the inequality $|x| + |y| \leq n$ has exactly $2n^2 + 2n + 1$ solutions where $x$ and $y$ are integers.

**Solution.** Let $P(n)$ be the proposition that there are $2n^2 + 2n + 1$ solutions to the inequality $|x| + |y| \leq n$, where $x$ and $y$ are integers. We prove that $P(n)$ holds for all $n \geq 0$ by induction on $n$.

In the base case, $P(0)$ is true because $2(0)^2 + 2(0) + 1 = 1$ and there is one solution to the equation $|x| + |y| \leq 0$, namely $x = y = 0$. This is easily argued by contradiction. Without loss of generality, suppose that $x$ is not zero. Then, since $|z| \geq 0$, for all $z$, and $|z| > 0$, for all $z \neq 0$, it follows that $|x| + |y| > 0$; a contradiction. Thus, $x$ must be zero.

In the inductive step, assume $P(n)$ to prove $P(n+1)$. That is, we want to prove that there are $2(n+1)^2 + 2(n+1) + 1 = 2n^2 + 6n + 5$ solutions to the inequality $|x| + |y| \leq n + 1$. Every solution to this inequality is also either a solution to the tighter inequality $|x| + |y| \leq n$ or else to the equation $|x| + |y| = n + 1$. Thus, the number of solutions is equal to the number of solutions to the tighter inequality plus the number of solutions to the equation.

By the inductive hypothesis, there are $2n^2 + 2n + 1$ solutions to the tighter inequality. All that remains is to count solutions to the equation. The value of $x$ must be in the range $-(n+1), \ldots, n+1$. If $x$ takes on either of the two values $-(n+1)$ or $n+1$, then there is a single solution, with $y = 0$. If $x$ takes on one of the $2n+1$ values in the range $-n, \ldots, n$, then there are two solutions, one with $y = n + 1 - |x|$ and one with $y = -(n + 1 - |x|)$. Thus, there are $2 + 2 \cdot (2n + 1) = 4n + 4$ solutions to the equation and $2n^2 + 2n + 1$ solutions to the tighter inequality, giving $2n^2 + 6n + 5$ solutions in all, as desired. Therefore, $P(n)$ implies $P(n+1)$, which concludes the inductive step.

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**Problem 4** The 6.042 mascot, Math Moose, made an amazing observation while playing with his collection of unit squares over the weekend.

First, Math Moose put his favorite unit square down on the floor as in diagram (a) of Figure 1. He noticed that the length of the periphery of the resulting shape was 4, an even number. Next, he put a second unit square down next to the first so that the two squares shared an edge as in diagram (b). He noticed that the length of the periphery of the resulting shape was now 6, which is also an even number. (The periphery of each shape in the figure is indicated by a thicker line.) Math Moose continued to place squares so that each new square shared an edge with at least one previously-placed square and no squares overlapped. Eventually, he arrived at the shape in diagram (c). He realized that the length of the periphery of this shape was 36, which is again an even number.

Math Moose spent the rest of the weekend trying to create a shape with an odd-length periphery, without any success. Help him out by proving by induction on the number of squares that the periphery length is always even, no matter how many squares he places or how he arranges them.

**Solution.** Let $P(n)$ be the predicate that the length of the periphery of shapes comprised of $n$ squares is even. We prove $P(n)$ for all $n \geq 1$ by induction on $n$. It is important to note that this predicate does not require the squares to be contiguous as discussed in the problem statement; that is, we have strengthened the predicate to include shapes comprised of a set of squares that are not necessarily contiguous. In the case of such shapes, the length of the periphery denotes the sum of the length of the periphery of all sub-shapes.

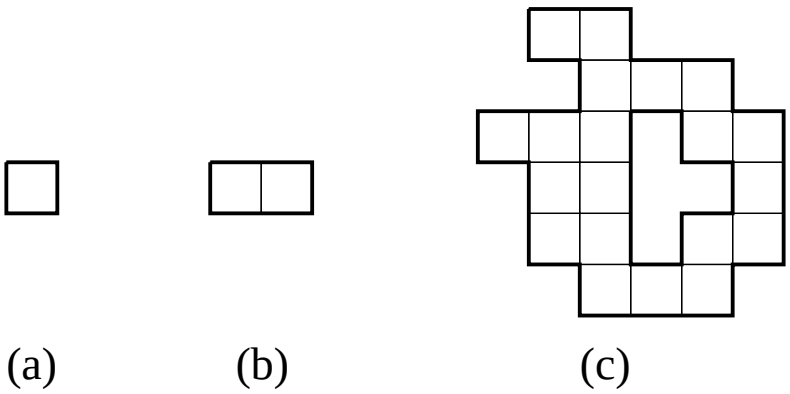


**Figure 1**: Some of the shapes that Math Moose created.

In the base case, $P(0)$ is true because the periphery of a shape comprised of 0 squares has length 0, which is even.

In the inductive step, we must show that for all $n \geq 0$, if the length of the periphery of the shapes comprised of $n$ squares is even, then the length of the periphery of shapes comprised of $n+1$ squares is even. Consider any collection of shapes comprised of $n+1$ squares; call this collection of shapes $S'$. Moreover, let $S$ denote collection of shapes obtained by removing an arbitrary square from $S'$; call this square $x$. By the inductive hypothesis, we know that the length of the periphery of $S$ is even. Now, consider the effect on the length of the periphery of adding the square $x$ to the shape $S$ so as to get $S'$. The square $x$ shares either 0, 1, 2, 3, or 4 edges with the squares comprising $S$. We analyze each of these cases separately:

- If the square $x$ shares no edges with any of the squares of $S$, then by adding the square $x$ the periphery length increases by four and thus remains even.
- If the square $x$ shares 1 edge with the squares of $S$, then this one edge is removed from the periphery, but three edges of the new square are added to the periphery. Overall, the periphery length increases by two and thus remains even.
- If the square $x$ shares 2 edges with the squares of $S$, then these two edges are removed from the periphery, but two edges of the new square are added. The periphery length is unchanged and thus remains even.
- If the square $x$ shares 3 edge with the squares of $S$, then these three edges are removed from the periphery, but one edge is added. The periphery length decreases by two and remains even.
- If the square $x$ shares 4 edge with the squares of $S$, then these four edges are removed from the periphery and none are added. The periphery length decreases by four and remains even.

In all cases, the length of the periphery remains even. Therefore, for all $n \geq 1$, $P(n)$ implies $P(n+1)$.

Thus, we have shown that the length of the periphery of any collection of shapes comprised of $n$ squares is even. As a special case of what we have shown, it follows that the length of the periphery of a shape of $n$ contiguous squares build by Math Moose is also even.

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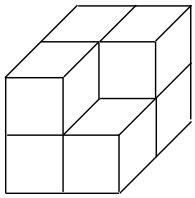
**Problem 5** The C.I.A. wants you to design plans to build, for an arbitrary $k \geq 1$, a $2^k \times 2^k \times 2^k$ cubic foot granite cube that is solid except for a missing single cubic foot that is one of the eight cubic feet nearest the center of the cube. It will have one of its corners exactly at the center of the large cube. The C.I.A. is not telling you what is to be placed inside the cube.

You are to build this by cementing together *basic blocks* that are 2-foot $\times$ 2-foot $\times$ 2-foot cubes, with one one-foot cube missing from its corner — as shown in the diagram below.



Describe your method for doing this. Your plans should be described recursively on $k$.

**Solution.** Let an $2^n \times 2^n \times 2^n$ cube with an arbitrary one-foot cubic block missing be denoted as an $n$-cube. Moreover, throughout this problem, we assume that the one-foot block that is missing from any $n$-cube is at a lattice point, *i.e.*, at an integral number of feet from each side.

**Theorem.** *For any positive integer $n$, any $n$-cube can be build out of 1-cubes.*

**Proof.** Let $P(n)$ be the predicate that an $n$-cube can be build out of 1-cubes. We prove that $P(n)$ holds for any positive integer $n$ by induction on $n$.

**Base Case:** $P(1)$ is true since a 1-cube is what we need to build. This is precisely what we are given as a basic block.

**Inductive Step:** We must show that for any positive integer $n$, $P(n)$ implies $P(n+1)$. Consider an $(n+1)$-cube. This block can be built using eight $n$-cubes as follows: build seven $n$-cubes such that they are all missing one of their corner blocks, arrange them so that their missing corner blocks meet in the center of the $(n+1)$-cube, fill up the gap with a 1-cube such that the corner corresponding to the eighth $n$-cube is left empty, and, finally, build the eighth $n$-cube such that the appropriate one-foot cubic block is missing. Notice, that the eight $n$-cubes can be build as desired because of the inductive hypothesis; that is, that any $n$-cube can be build out of 1-cubes.

■

Now, we can state the following corollary.

**Corollary.** *For any positive integer $n$, an $n$-cube with any one-foot cubic block missing from its center can be build out of 1-cubes.*

**Proof.** $n$-cubes of this particular form, *i.e.*, with any one-foot cubic block missing from their center, is a particular case of what we showed in the above Theorem. Thus, we know that $n$-cube with any one-foot cubic block missing from its center can be build out of 1-cubes. The construction of such a block can be done as outlined in the above theorem, with the exception that the missing one-foot block is missing from the center of the $(n+1)$-cube. ■

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**Problem 6** One plan considered by the 6.042 staff, then dropped, involved requiring students in tutorials to work in groups of either exactly 4 or exactly 7 students. We abandoned this plan when we realized that a tutorial section of size 13 could not be split up in this way.

**(a)** What is the smallest number $k$ such that all tutorials of $k$ or more students *can* be split into teams of 4 and 7?

**(b)** Prove that $k$ is the smallest such number.

**(c)** Prove (by induction) that all tutorials of $k$ or more students can be split into teams of size 4 and 7.

**Solution.**

**(a)** The smallest number $k$ such that all tutorials of $k$ or more students *can* be split into teams of 4 and 7 is 18.

**(b)** A tutorial of size 17 *can not* be split into teams of 4 and 7. In particular, a tutorial of size 17 can contain at most 2 groups of 7. Consider the various possible splits of the tutorial: i) no groups of 7: since the only other group size is 4 we can not split the tutorial of 17; ii) one group of 7: the remaining 10 students can not be split into groups of 4; iii) two groups of 7: the remaining 3 students can not be split into groups of 7.

**(c)** The proof is by strong induction on the size $n$ of the tutorial. Let $P(n)$ be the predicate that a tutorial of size $n$ can be split into groups of size 4 and 7.

We must show that $P(n)$ holds for all $n \geq 18$.

1. **Base Case:** $P(18)$, $18 = 4 + 7 + 7$
2. **Base Case:** $P(19)$, $19 = 4 + 4 + 4 + 7$
3. **Base Case:** $P(20)$, $20 = 4 + 4 + 4 + 4 + 4$
4. **Base Case:** $P(21)$, $21 = 7 + 7 + 7$
5. **Inductive Step:** $\forall\, n \geq 21, (\forall\, k \in [18, n], P(k)) \implies P(n+1)$
    1. Fix $n \geq 21$.
    2. From the inductive hypothesis $P(k)$ holds, for all $k \in [18, n]$.
    3. In particular, $P(n-3)$ holds.
    4. The size of the $n+1$ student tutorial is 4 students greater than the $n-3$ student tutorial.
    5. Thus, we can split up the $n+1$ tutorial as the $n-3$ tutorial
       and with an extra 4 student group.
    6. QED
6. QED

□

**Problem 7** Let the sequence $G_0, G_1, G_2, \ldots$ be defined recursively as follows: $G_0 = 0$, $G_1 = 1$, and $G_n = 5G_{n-1} - 6G_{n-2}$, for every $n \in \mathbb{N}, n \geq 2$.

Prove that for all $n \in \mathbb{N}$, $G_n = 3^n - 2^n$.

**Solution.** Let $P(n)$ be the predicate that $G_n = 3^n - 2^n$. We prove $P(n)$, for all $n \in \mathbb{N}$, by strong induction on $n$.

**Base Case:**

In order to simplify the strong induction, we use two base cases, in particular $n = 0$ and $n = 1$. By the definition of the sequence it is the case that $G_0 = 0$. For $n = 0$, it is the case that $3^n - 2^n = 3^0 - 2^0 = 1 - 1 = 0$; that is, it follows that $P(0)$ holds. Similarly, by the definition of the sequence it is the case that $G_1 = 1$. For $n = 1$, it is the case that $3^n - 2^n = 3^1 - 2^1 = 3 - 2 = 1$; that is, it follows that $P(1)$ holds.

**Inductive Step:**

We must show that for all $n \in \mathbb{N}, n \geq 2$, it is the case that $(\forall\, k \in [0, n], P(k)) \implies P(n+1)$.

1. Fix $n \in \mathbb{N}, n \geq 2$.
2. Assume that the elements of the sequence $G_x$, for $0 \leq x \leq n$, can be expressed as $G_x = 3^x - 2^x$.
3. The element of the sequence $G_{n+1}$ can be expressed as $G_{n+1} = 3^{n+1} - 2^{n+1}$.
   (a) By the definition of the sequence we have $G_{n+1} = 5G_n - 6G_{n-1}$.
   (b) By our induction hypothesis, it is the case that $G_n = 3^n - 2^n$ and $G_{n-1} = 3^{n-1} - 2^{n-1}$
   (c) Thus, $G_{n+1} = 5(3^n - 2^n) - 6(3^{n-1} - 2^{n-1})$. Working through the algebra, we conclude that $G_{n+1} = 3^{n+1} - 2^{n+1}$, as needed for the inductive step.
   (d) QED
4. QED (implication and UG)

QED (induction)

□

**Problem 8** Construct the following rankings for $n$-player tournaments, for any $n \geq 2$, or show that this is not always possible.

**(a)** Assume that one player, Enrique, wins all his matches, and another, Ricky, wins all but one of his matches. Construct a ranking that is locally fair, and in which Enrique is first and Ricky second.

**(b)** Assume that there are no sets of 3 players, A, B, and C, such that A beats B, B beats C, and C beats A; that is, $\nexists A, B, C, [(A \text{ beats } B) \wedge (B \text{ beats } C) \wedge (C \text{ beats } A)]$. Construct a ranking that is extremely fair, in the sense that any player X that beats another player Y appears higher than Y in the rankings.

**Solution.**

**(a)** Since Enrique beat everyone and Ricky beat everyone but one person, it follows that Ricky lost to Enrique. Now, consider all the other players in the tournament. As seen in class, they can be ordered in a locally-fair ranking. It follows that a locally-fair ranking for all the players is the ranking with Enrique first, Ricky second, and the remaining players ranked based on a locally-fair ranking. We know that Enrique beat Ricky, that Ricky beat the player ranked third since he beat everyone but Enrique, and that the remaining players are ranked in a locally-fair fashion.

**(b)**

**Theorem.** *In any tournament with no 3-player cycles, a locally-fair ranking is extremely fair.*

**Proof.** First, consider the case when there is only one or two players. In the case of one player, the ranking is trivial. In the case of two players, an extremely fair ranking can be constructed by placing the winner first and the loser second. Now consider a tournament of at least 3 players, in which there are no 3-player cycles. In class we have shown that if a tournament has a cycle, then the tournament has a 3-player cycle. Thus, the contrapositive is also true; that is, if a tournament has no 3-player cycle, then the tournament has no cycles.

Consider a locally-fair ranking of a tournament with no 3-player cycles. We show that the locally-fair ranking is extremely fair by contradiction. Assume that the locally-fair ranking is not extremely fair. Since the ranking is not extremely fair, there exist players $X$ and $Y$ such that $X$ appears higher in the ranking than $Y$ even though $Y$ beat $X$, *i.e.*, $\cdots X \rightarrow \cdots \rightarrow Y \cdots$ and $Y \rightarrow X$. It follows that the tournament graph contains the cycle $X \rightarrow \cdots \rightarrow Y \rightarrow X$. Since it contains a cycle, it must also contain a 3-player cycle. This contradicts the fact that the tournament has no 3-player cycles. It follows that our assumption that the ranking is not extremely fair is false. ■

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**Problem 9** The set $W$ of "well-formed parenthesis strings" can be defined by the following rules:

1. the empty string is in $W$, and
2. if $s$ and $t$ are in W then the string $(s)t$ is in $W$.

For example, the string $(()())()$ is a well-formed parenthesis string, where as the string $)(())$, isn't.

Use structural induction to prove that any prefix of any well-formed parenthesis string has at least as many left parentheses as right parentheses.

**Solution.** Let $P(t)$, for $t \in W$, be the predicate that any prefix of a well-formed parenthesis string $t$ has at least as many left as right parentheses. We prove that $P(t)$ holds for all $t \in W$ by structural induction.

**Base Case:**

Let $t$ be the empty string $\lambda$. $P(\lambda)$ holds since there are no left or right parentheses.

**Inductive Step:**

We must show that for all $s, t \in W$, $P(s)$ and $P(t)$ imply $P((s)t)$; that is, that any prefix of the string $(s)t$ has at least as many left as right parentheses. Let $s, t$ be any strings in $W$ and assume that $P(s)$ and $P(t)$ hold. Moreover, for any $s_1$ and $s_2$, let $s_1 \leq s_2$ denote that $s_1$ is a prefix of the string $s_2$. Let $p$ be any prefix of the string $(s)t$ and consider the following set of exhaustive cases:

$p = \lambda$

In this case, $p$ trivially satisfies the claim.

$p = ($

In this case, $p$ trivially satisfies the claim.

$p = (s'$**, for some string** $s'$**,** $s' \leq s$

In this case, since $P(s)$ holds, it follows that $s'$ has at least as many left as right parentheses. Thus, the same property holds for the string $(s'$.

$p = (s)$

In this case, since $P(s)$ holds, the string $(s)$ has at least as many left as right parentheses.

$p = (s)t'$**, for some string** $t'$**,** $t' \leq t$

In this case, by the previous case, we know that the string $(s)$ has at least as many left as right parentheses. Moreover, since $P(t)$ holds, it follows that $t'$ has at least as many left as right parentheses. Thus, it follows that the string $(s)t'$ has at least as many left as right parentheses.

Thus, any prefix of $(s)t$ has at least as many left as right parentheses; that is, $P((s)t)$ holds. Finally, by universal generalization, the inductive step holds for any such $s, t \in W$.

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