Quiz 2 Solutions

Problem 1 [10 points]

Consider $n > 1$ people that attend a party. Prove that there must be two people who greet the same number of distinct people; assuming no one greets his/her self.

Solution. A straightforward application of the pigeonhole principle. Since no person can greet himself, the number of people he can greet must be an integer from 0 to $n-1$. If no two people greet the same number of people, then every one of these n numbers from 0 to $n-1$ must equal the number of greetings some person made. But that would mean that some person greets 0 people while some other person greets $n-1$ people, i.e., all the other people. This is a contradiction.

Problem 2 [20 points] **Counting**

For the problems below, you may leave products and binomial coefficients in your final answer. Moreover, you should assume that a deck of cards consists of 52 cards; that is, 4 suits of 13 cards each $(2, 3, 4, \dots, 9, 10, J, Q, K, A)$.

(a) [5 points] In poker, a *flush* is a set of five cards, all of the same suit. How many distinct flushes are there?

Solution. $4 \cdot \binom{13}{5}$: 4 for the suit, $\binom{13}{5}$ for the cards chosen within that suit.

(b) [5 points] In poker, a *straight* is a set of five cards that, when the suits are ignored, form a five-in-a-row sequence, for example $5, 6, 7, 8, 9$ or $8, 9, 10, J, Q$. An ace can start a straight $(A, 2, 3, 4, 5)$ or end it $(10, J, Q, K, A)$ but cannot wrap around (**no** $Q, K, A, 2, 3$). How many distinct straights are there?

Solution. $4^5 \cdot 10$: 4 suits for each card, and we can start the straight at any one of 10 cards (A through 10).

(c) [5 points] A *straight flush* is a set of five cards that is both a straight and a flush: 5 cards in a row, all the same suit. How many are there?

Solution. $4 \cdot 10$

(d) [5 points] How many 5-card sets are either a straight *or* a flush (or both)?

Solution. Inclusion-exclusion: $4^5 \cdot 10 + 4\binom{13}{5} - 4 \cdot 10$

Problem 3 [10 points] **Counting**

Consider placing n distinguishable books on r distinguishable shelves.

(a) [5 points] If the order of books on each shelf doesn't matter, how many distinct placements are there?

Solution. Product rule: r shelves for each of n books, so r^n.

(b) [5 points] If order on the shelf matters, how many distinct placements are there?

Solution. For the first book there are r choices. For the next book there are $r + 1$ choices since we can think of the first book as splitting the shelf it is put on into two shelves. Similarly for the third book there are $r+2$ choices. In this way we see that the total number of ways, by the product rule, is $r \times (r + 1) \times (r + 2) \ldots (n + r - 1)$.

Problem 4 [10 points] **Order of Growth**

(a) [3 points] From lecture, we know that any comparison-based sorting algorithm $\mathbb{A}$ must take $\Omega(n \log n)$ steps on n elements. Which one of the following is a consequence of this fact?

a) There is some constant $c > 0$ for which all executions of $\mathbb{A}$ will perform exactly $cn \log n$ steps.

b) There is some constant $c > 0$ for which all executions of $\mathbb{A}$ will perform at least $cn \log n$ steps.

c) There is some constant $c > 0$ for which some executions of $\mathbb{A}$ will perform at least $cn \log n$ steps.

d) There is some constant $c > 0$ for which all executions of $\mathbb{A}$ will perform at most $cn \log n$ steps.

e) There is some constant $c > 0$ for which some executions of $\mathbb{A}$ will perform at most $cn \log n$ steps.

Solution. c is the correct answer. Ω-notation deals only with worst-case lower bounds; $\Omega(n \log n)$ is a lower bound on the number of steps required for the *worst-case* input to $\mathbb{A}$.

(b) [3 points] Prove that $10n^2 = O(n^3)$.

Solution. $10n^2 \leq c \cdot n^3$ for all $n \geq 1$ when $c \geq 10$, so the theorem follows.

(c) [4 points] Prove that n^3 is not $O(10n^2)$.

Solution. Assume $\exists c$ such that for all $n > N$, $n^3 < c \cdot 10n^2$. But this implies that $n^3/n^2 = n < 10c = c \cdot 10n^2/n^2$, which is not true for any $n \geq 10c$, so it is not true for all $n > N$. This is a contradiction.

Problem 5 [10 points] **Summations**

(a) [5 points] Determine the closed-form expression for the summation:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} x^{k+j}y^{k-j}$$

Solution. Rearrange:

$$\begin{aligned}
\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} x^{k+j}y^{k-j} &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} x^k x^j y^k y^{-j} \\
&= \sum_{k=0}^{\infty} x^k y^k \sum_{j=0}^{\infty} x^j y^{-j} \\
&= \left(\sum_{k=0}^{\infty}(xy)^k\right)\left(\sum_{j=0}^{\infty}\left(\frac{x}{y}\right)^j\right) \\
&= \frac{1}{1-xy}\cdot\frac{1}{1-x/y}
\end{aligned}$$

when $|xy| < 1$ and $|x/y| < 1$.

(b) [5 points] Prove a tight asymptotic bound (Θ) for the expression:

$$\sum_{k=2}^{n}\frac{1}{k\ln k}$$

Hint: $\dfrac{1}{x\ln x} = \dfrac{d}{dx}\ln\ln x$

Solution. $\sum_{k=2}^{n}\frac{1}{k\ln k} = \sum_{k=2}^{n}\frac{d}{dk}\ln\ln k$, so we use the integral method:

$$\begin{array}{rcl}
\int_2^n \frac{d}{dk}\ln\ln k\ dk \leq & \sum_{k=2}^{n}\frac{d}{dk}\ln\ln k & \leq \int_2^n \frac{d}{dk}\ln\ln(k+1)\ dk \\
\ln\ln k|_2^n \leq & & \leq \ln\ln(k+1)|_2^n \\
\ln\ln n - \ln\ln 2 \leq & & \leq \ln\ln(n+1) - \ln\ln 3 \\
\ln\ln n - \ln\ln 2 \leq & & \leq \ln\ln(n+1)
\end{array}$$

In the limit as $n \to \infty$, the left term is dominated by $\ln\ln n$, so it is lower bounded by $c\ln\ln n$ for some constant $0 < c < 1$. Thus, the center is $\Omega(\ln\ln n)$ and $O(\ln\ln n)$.

Problem 6 [10 points] **Binomial Coefficients**

Prove that

$$\sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}$$

(a) [5 points] using a combinatorial argument. **Hint:** Consider choosing a committee that is headed by a leader.

Solution. Consider the number of ways of picking a team of size k from a group of n people and designating one of them captain. This can be done in $k\binom{n}{k}$ ways, for all $k = 0, 1, \ldots, n$. But the total number of ways of doing this can also be counted differently. First select the captain - n ways - then select an arbitrary subset of the rest - 2^{n-1} ways. Hence the result follows.

(b) [5 points] by finding a closed-form expression for $\sum_{k=1}^{n} \binom{n}{k} kx^k$ and evaluating it at $x = 1$.

Solution. Differentiating

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

we get

$$n(1+x)^{n-1} = \sum_{k=1}^{n} k \binom{n}{k} x^{k-1}$$

and evaluating it at $x = 1$ we get the desired result.

Problem 7 [10 points] **Recurrences**

Consider the following recurrence to which the master theorem *does not apply:*

$$T(n) = 2T(n/2) + n \log_2 n$$

(a) [3 points] Perform two iterations of plug-and-chug to reveal a pattern for this recurrence.

Solution. See below.

(b) [7 points] Determine a tight asymptotic bound (Θ) for this recurrence. You may assume n is a power of two if it is convenient.

Solution. The master theorem does not apply (c.f. CLR, p.63). So, we expand:

$$\begin{aligned}
T(n) &= 2T(n/2) + n \log n \\
&= 2^2 T(n/2^2) + n \log \frac{n}{2} + n \log n \\
&= 2^3 T(n/2^3) + n \log \frac{n}{2^2} + n \log \frac{n}{2} + n \log n \\
&\vdots \\
&= 2^{\log n} T(n/2^{\log n}) + n \left(\log \frac{n}{2^{\log n - 1}} + \cdots + \log \frac{n}{2^2} + \log \frac{n}{2} + \log n \right) \\
&= \Theta(n) + n \left(\log 2 + \log 4 + \cdots + \cdots + \log \frac{n}{2} + \log n \right) \\
&= \Theta(n) + n \left(1 + 2 + \cdots + \log n \right) \\
&= \Theta(n) + n \frac{(\log n)(1 + \log n)}{2} \\
&= \Theta(n) + \Theta(n \log^2 n) \\
&= \Theta(n \log^2 n)
\end{aligned}$$

Problem 8 [10 points] **Recurrences**

The Millicent computer virus has been spreading on computers around MIT. Once a copy of Millicent enters a computer, it is copied to 2 new (uninfected) computers after one day, then to 3 new computers the second day, and then detected and removed on the third day. Millicent's creator installed Millicent on 2 computers on day 0 and 2 additional computers on day 1.

(a) [3 points] Due to a tip from inside the FBI that Millicent might be of extraterrestrial origin, Frohike and Langley have been investigating the growth of the Millicent virus; however, they've gotten stuck. Help them by stating a recurrence describing the number of computers **newly** infected on day n for $n > 1$.

Solution. $T_n = 2T_{n-1} + 3T_{n-2}$

(b) [4 points] Find a closed form for the recurrence you stated in the previous part.

Solution. The characteristic equation is $x^2 - 2x - 3 = 0$, which has roots at 3 and -1. The solution is then of the form

$$T_n = A3^n + B(-1)^n$$

We plug in the given initial conditions:

$$\begin{aligned} T_0 = 2 &= A + B \\ T_1 = 2 &= 3A - B \end{aligned}$$

Solving this system yields the coefficients $A = B = 1$. So, the closed form for this recurrence is simply $T_n = 3^n + (-1)^n$.

(c) [3 points] Fearing the efforts of the mighty Frohike and suave Langley, Millicent's creator became frightened and decided to start installing the virus on 4 new computers every day after day 1. State a recurrence for the revised problem and find a closed form solution for the number of computers newly infected on day n.

Solution. The general solution to the homogeneous recurrence is of the form

$$T_n = A3^n + B(-1)^n$$

as above. Now we need to find a particular solution to the inhomogeneous recurrence. Since the inhomogeneous term is constant, we guess a constant solution b:

$$b = 2b + 3b + 5 \Longrightarrow b = -1$$

Therefore, the inhomogeneous recurrence has the simple particular solution $T_n = -1$. We now need to determine the constants on the complete solution

$$T_n = A3^n + B(-1)^n - 1$$

by plugging in the initial conditions and solving the resulting system:

$$\begin{aligned} T_0 = 2 &= A + B - 1 \\ T_1 = 2 &= 3A - B - 1 \end{aligned}$$

which has the solution $A = B = 3/2$. The closed form for this recurrence is then

$$T_n = \frac{3}{2}3^n + \frac{3}{2}(-1)^n - 1$$

We can perform a sanity check to make us more confident of our answer:

$$\begin{array}{l} T_2 = 2 \cdot 2 + 3 \cdot 2 + 4 = 14 \\ T_2 = (3/2)3^2 + (3/2)(-1)^2 - 1 = 14 \\ \hline T_3 = 2 \cdot 14 + 3 \cdot 2 + 4 = 38 \\ T_3 = (3/2)3^3 + (3/2)(-1)^3 - 1 = 38 \end{array}$$

Problem 9 [10 points] **Countability**

(a) [5 points] A function $f : \mathbb{N} \to \mathbb{N}$ is *strictly increasing* if $f(n+1) > f(n)$ for all n. Prove that the set of such strictly increasing functions is uncountable. **Hint:** Use diagonalization.

Solution.

To use diagonalization, we must assume that the set is countable, then find a contradiction by performing two steps:

1. Describe an **arbitrary** enumeration (ordered listing) of **all** the elements in the set.
2. Construct an item that is **in the set** but **not equal** to any element of the enumeration.

The emphasized elements are the crucial components; most false diagonalization proofs fail to satisfy one of those conditions.

The simplest diagonalization may be the following: List the functions in some arbitrary enumeration F, where F_i is the i^{th} element of the list. Now, for any $F_i(x)$, the next value of the function, $F_i(x+1)$, is either 1 greater than $F_i(x)$ or more than 1 greater than $F_i(x)$. We can now make a new function, F_d, where $F_d(x) = F_d(x-1)+1$ if $F_x(x) - F_x(x-1) > 1$ and $F_d(x-1)+2$ otherwise. F_d must be in the set, since it is strictly increasing. However, F_d cannot be equal to any F_i, since it does not increase by the same amount from its value at $i-1$ to its value at i.

The proof could also have been done by making F_d increase by 1 more than $F_x(x) - F_x(x)$ from $F_d(x-1)$ to $F_d(x)$. We could even construct $F_d(x)$ as $F_x(x) + F_d(x-1)$.

In fact, an even simpler proof could have been to merely notice that the above analysis suggests a surjection from the set of increasing functions to the set of infinite binary strings, since every increase may either be 1 or more than 1:

$$\begin{array}{lcllllll} F_0 & : & 2 & 3 & 7 & 8 & 9 & \ldots \\ F_1 & : & 6 & 7 & 10 & 13 & 22 & \ldots \\ F_2 & : & 2 & 4 & 8 & 9 & 16 & \ldots \\ F_3 & : & 1 & 5 & 6 & 7 & 40 & \ldots \end{array} \tag{1}$$

maps to:

$$\begin{array}{lclllll} F'_0 & : & 0 & 1 & 0 & 0 & \ldots \\ F'_1 & : & 0 & 1 & 1 & 1 & \ldots \\ F'_2 & : & 1 & 1 & 0 & 1 & \ldots \\ F'_3 & : & 1 & 0 & 0 & 1 & \ldots \end{array} \tag{2}$$

This is an injection to the set of infinite bit strings, which we have shown to be uncountable. (We could also have made a surjection from increasing functions to decimal expansions of reals, with the n^{th} digit of r_i being $F_i(n) \bmod 10$...)

(b) [5 points] A function $f : \mathbb{N} \to \mathbb{N}$ is *strictly decreasing* if $f(n+1) < f(n)$, for all n such that $f(n) > 0$, and $f(n+1) = 0$, for all n such that $f(n) = 0$. For example, the function f could be $f(0) = 5, f(1) = 3, f(2) = 0, f(3) = 0, \cdots$.

Prove that the set of such strictly decreasing functions is countable. **Hint:** First prove that, for any $k \in \mathbb{N}$, the set of strictly decreasing functions for which $f(0) = k$ is countable.

Solution. Suppose $f(0) = k$. Then, since $f(x)$ decreases by at least 1 for each increment of x, $f(k)$ must be 0. By the definition, $f(x) = 0$ for all $x > k$.

We can describe each f where $f(0) = k$ by noting which numbers from 1 to k are in the range. (It can never be larger than k, because the function is decreasing). In fact, this uniquely defines each of the functions. We can view that as a finite bitstring of length k, where position i is 1 if i is in the range and 0 otherwise. Because k can have any value, the decreasing functions map to finite bitstrings of all lengths. This is thus an injection into the set of finite bitstrings. (Actually, a bijection, if we don't include the bit for k itself). Since we have proven before that the set of finite bitstrings is countable, the set of decreasing functions must also be countable.

We could have instead shown that functions of a given starting k, there are only 2^{k-1} possible different functions, corresponding to whether each number from 1 to $k-1$ is in the range or not. Thus, there are a finite number of decreasing functions for each starting k. Since k can only be any natural number, there are a countable number of such sets comprising all of the decreasing functions. A union of countably-many finite sets is countable.

We could even have made a direct injection into the natural numbers or finite bitstrings by mapping each f to $f(0)$ 1's followed by 0 followed by $f(1)$ 1's followed by 0 followed by $f(2)$ 1's followed by 0 followed by $\ldots$ for all $f(i) > 0$.