Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
Professors David Karger and Nancy Lynch

Handout 32
April 6, 2000

Practice Quiz 2

Due Date: Never.

Problem 1

(a) Give tight (Θ) bounds for

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}}.$$

(b) Express the following in closed form

$$\sum_{p=1}^{\infty} \sum_{q=p}^{\infty} x^p y^q.$$

(Note that the lower indices are not constants.)

Problem 2 I have n one-dollar bills that I wish to donate to m charities.

(a) In how many ways can I do this if I don't care how much each charity gets?

(b) In how many ways can I do this so that each charity gets at least \$ 10?

Problem 3 In how many ways can 5 indistinguishable covered wagons, 4 indistinguishable uncovered wagons and 1 horse be arranged in a circle?

Problem 4 How many 6-digit decimal numbers are there that do not contain "123" or "456" as a subsequence? (for purposes of this problem numbers may have leading 0's)

Problem 5 Consider the following nuclear reaction. There is one particle of type 1 at time instant 1. Each particle of type i existing at time t undergoes fission to produce one particle of type $i+1$ and one of type $i+2$ at time instant $t+1$.

(a) Write and justify a recurrence for $f(i,t)$, the number of particles of type i at time t. (Make sure to include the base cases for your recurrence.)

(b) Prove by induction that $f(i,t) = \binom{t-1}{i-t}$. (You do not need to reprove identities that were proven in class.)

Problem 6

(a) Give tight (Θ) bounds for $T(n)$. Assume that $T(n)$ is constant for $n \leq 10$.

$$T(n) = 4T(n/2) + 6n.$$

(b) Rank the following functions by order of growth; that is find an arrangement $g_1 = \alpha(g_2) = \alpha(g_3) = \ldots = \alpha(g_6)$ and state whether α is o or Θ for each step.

- $n(\log_2 n)^5$
- n^3
- $n!$
- $(n!)^2$
- n^n
- $3 \cdot 8^{\log_2 n} + 4^{\log_2 n}$

Problem 7 Prove or disprove: $f(n) = O(g(n))$ implies $\lg(f(n) = O(\lg(g(n)))$, where $g(n)$ is nondecreasing, $g(1) > 1$ and $f(n) \geq 0$ for all sufficiently large n.

Problem 8 Solve the following recurrence:

$$f(1) = f(2) = 1$$

$$f(n) - 4(n-1) + 4(n-2) = 3^n$$

Problem 9 Give tight (Θ) bounds for $T(n)$. Assume that $T(n)$ is constant for $n \leq 2$.

(a) $T(n) = 7T(n/3) + n^2$

(b) $T(n) = T(2n/3) + T(n/4) + n$

(c) $T(n) = 4T(n/3) + n$

Problem 10 Pick any set A of ten numbers from 1 to 100. Prove that there must exist two distinct, non-empty subsets whose sums are equal. E.g., consider the set $A = \{51, 11, 81, 68, 73, 87, 23, 29, 25, 94\}$; in this case, the subsets $\{25, 51, 29\}$ and $\{94, 11\}$ both add to 105. *Hint:* Consider the number of subsets of a 10-element set.