Quiz 1

Your Name: ______________________________

Circle the name of your Tutorial Instructor:

Carl	David	Josh	Ken
Kyle	Robert	Shishir	Victor

Write the hour at which your Friday recitation begins ______

- This quiz is open book/notes.
- There are 11 pages with 7 problems totaling 70 points. Problem parts are labeled with their point values.
- Put your name on the top of **every** page — *these pages may be separated for grading.*
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- You may assume any of the results presented in class or in the lecture notes.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. **Be neat and write legibly**.
- GOOD LUCK!

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	10		
2	10		
3	10		
4	10		
5	10		
6	5		
7	15		
Total	70		

Problem 1 [10 points] **Quantifiers**

Consider the following predicates where the universe of discourse is the set of all people:

$P(x)$: x is a hacker

$Q(x)$: x is an MIT student

$R(x)$: x is an artist

$S(x)$: x is willing to waltz

Express each of the following statements using quantifiers, logical connectives, and and the above predicates:

(a) [1 point] No hackers are willing to waltz.

Solution. $\nexists x \;\; P(x) \wedge S(x)$

(b) [1 point] No artists are unwilling to waltz.

Solution. $\nexists x \;\; R(x) \wedge \neg S(x)$

(c) [1 point] All MIT students are hackers.

Solution. $\forall x \;\; Q(x) \Rightarrow P(x)$

(d) [1 point] MIT students are not artists.

Solution. $\forall x \;\; Q(x) \Rightarrow \neg R(x)$

(e) [4 points] Does (d) follow logically from (a), (b), and (c)? If so, give a proof. If not, give a counterexample.

Solution. Assume (a), (b), and (c). Then, we want to prove that, for any x such that $Q(x)$ (for any MIT student), it follows that $\neg R(x)$ (that student is not an artist).

From (c), we know that $P(x)$ (that the student is a hacker). If we rearrange (a) to

$$\forall x \;\; P(x) \Rightarrow \neg S(x)$$

then clearly $\neg S(x)$ (the student is not willing to waltz). So, if we rearrange (b) to

$$\forall x \;\; \neg S(x) \Rightarrow \neg R(x)$$

it follows that $\neg R(x)$ (the student is not an artist).

(f) [1 point] Translate into English: $(\forall x)(R(x) \vee S(x) \implies Q(x))$.

Solution. All who are artists or people willing to waltz are MIT students.

(g) [1 point] Translate into English: $(\exists x)(R(x) \wedge \neg Q(x)) \implies (\forall x)(P(x) \implies S(x))$.

Solution. If there is an artist who is not an MIT student, then all hackers are willing to waltz.

Problem 2 [10 points] **Induction**

Prove **by induction** (other methods will not receive credit) that

$$(1 \cdot 2) + (2 \cdot 3) + (3 \cdot 4) + \cdots + (n-1)n = \frac{1}{3}(n-1) \cdot n \cdot (n+1)$$

whenever n is a natural number greater than 1.

Solution. Prove $P(n) ::= (1 \cdot 2) + (2 \cdot 3) + (3 \cdot 4) + \cdots + (n-1)n = \frac{1}{3}(n-1) \cdot n \cdot (n+1)$

Basis. $(P(2))$ $(1 \cdot 2) = \frac{1}{3}(1) \cdot 2 \cdot (3)$.

Induction. Assume $P(n)$. Prove $P(n+1)$.

$$\begin{aligned}
\underbrace{(1 \cdot 2) + (2 \cdot 3) + (3 \cdot 4) + \cdots + (n-1)n}_{\text{Apply IH}} + n(n+1) &= \frac{1}{3}(n-1) \cdot n \cdot (n+1) + n(n+1) \\
&= n(n+1)\left[\frac{1}{3}(n-1) + 1\right] \\
&= n(n+1)\left[\frac{1}{3}n + \frac{2}{3}\right] \\
&= \frac{1}{3}n(n+1)\left[n+2\right]
\end{aligned}$$

Problem 3 [10 points] **Structural Induction**

Consider the sets $S_1, S_2, \ldots$ defined inductively as

- $S_0 = \{0, 1\}$
- For all $n \geq 1$, $S_n = \left\{\frac{x+y}{2} \mid x, y \in S_{n-1}\right\}$

i.e., where S_n contains the average values of each pair of numbers in S_{n-1}. Note we allow $x = y$ in the definition of S_n.

(a) [1 point] Write out the elements of each of the following sets: S_1, S_2, and S_3.

Solution.

$$\begin{array}{ll}
S_1 & \{0, 1/2, 1\} \\
S_2 & \{0, 1/4, 1/2, 3/4, 1\} \\
S_3 & \{0, 1/8, 1/4, 3/8, 1/2, 5/8, 3/4, 7/8, 1\}
\end{array}$$

(b) [2 points] Generalize (in your own words or in symbols) to give the elements of S_n.

Solution. S_n contains $2^n + 1$ elements (including 0 and 1) evenly spaced along the unit interval; i.e., $S_n = \{i/2^n | 0 \leq i \leq 2^n\}$.

(c) [7 points] Prove by induction that for all $n \in \mathbb{N}$, $1/3 \notin S_n$. You will probably want to strengthen the induction hypothesis. You may not assume your generalization from the previous part is true; you must prove that statement if you wish to use it.

Solution. Strengthen the theorem to

$$P(n) = \text{"For all } a/b \in S_n \text{ with } a, b \in \mathbb{N} \text{ where } a/b \text{ is in reduced form, } b = 2^k \text{ for some } k\text{"}$$

Clearly, if $\forall n\ P(n)$, then $\forall n\ 1/3 \notin S_n$. Thus, we need only prove $\forall n\ P(n)$.

Base cases. $0 = 0/2^0$, $1 = 1/2^0$.

Induction. Assume $P(n)$. By the construction of S_{n+1}, all elements of S_{n+1} are of the form $(x + y)/2$ for some $x, y \in S_n$. Fix any two such elements $x, y \in S_n$, where (by the induction hypothesis) $x = a/2^k$ and $y = c/2^\ell$ with $a, b, k, \ell \in \mathbb{N}$.

Assume WLOG $k \geq \ell$. Then,

$$\begin{aligned}
(x + y)/2 &= (a/2^k + c/2^\ell)/2 \\
&= (a/2^k + c2^{k-\ell}/2^k)/2 \\
&= (a + c2^{k-\ell})/2^{k+1}
\end{aligned}$$

While this may not be in reduced form, reduction cannot introduce any non-2 factors into the denominator, so this fits the form specified by the theorem. Therefore, $P(n + 1)$.

By induction, $\forall n\ P(n)$. Thus, $\forall n\ 1/3 \notin S_n$.

Problem 4 [10 points] **Relations**

(a) [1 point] What is the common name for the reflexive closure of the "less than" relation?

Solution. less than or equal to

(b) [1 point] What is the common name for the transitive closure of the "child of" relation?

Solution. descendant of

(c) [2 points] What is the common name for the inverse of the "teaches" relation (that is, if xTy means x teaches y, how do we say $aT^{-1}b$)?

Solution. is taught by

(d) [3 points] Consider the partial order "aRb iff a divides b without remainder" on the universe of natural numbers **greater than 1**. What are the minimal elements under this relation usually called?

Solution. primes

(e) [3 points] Is the relation "xRy iff $x = y$ or x is **not** y's roommate" an equivalence relation? Why or why not?

Solution. No, because it is not transitive. If x and y are roommates but neither is a roommate of z, then xRz and zRy but $\neg xRy$.

Problem 5 [10 points] **Precedence Relations**

Death is planning the end of the world. This involves a number of tasks each of which takes one minute to complete. The prerequisites associated with these tasks are listed below.

ABBRV.	TASK	PREREQUISITES
C	Compose a requiem	
N	Notify the UN	B
D	Signal the daemons	B
B	Blow the trumpet	
T	Sell T-shirts: "All I got was this lousy t-shirt."	N
Q	Grade the 6.042 quiz	S
G	Open the gates	D,N
S	Put out the sun	C,N

(a) [5 points] Represent the tasks and their prerequisites as a directed graph.

Solution. This is very similar to the constructions in the notes.

(b) [5 points] Death is omnipotent and can therefore work on as many tasks at a time as he wishes. What is the minimum amount of time required for him to end the world? Why? (An explanation without proof is fine.)

Solution. 4 time units, since that is the length of the critical path.

Problem 6 [5 points] **Graphs**

Let $G = (V, E)$ be a simple, undirected graph, and let C be a set of colors. Define a *partial coloring of G using C* to be a function that assigns to each node in V either a color in C or no color. (That is, it colors some, none, or all of the nodes using colors in C.) Define a *partly colored graph* to be a simple, undirected graph $G = (V, E)$ together with a partial coloring of G.

Prove the following fact about partly colored graphs:

In a partly colored graph, any walk connecting a colored node to an uncolored node must include a colored node adjacent to an uncolored node.

Solution. Consider the path $p = v_1 v_2 \cdots v_k$, of which v_1 is colored and v_k is uncolored. Assume there is no edge (v_i, v_{i+1}) such that v_i is colored and v_{i+1} is uncolored. Then, by induction starting from v_1, v_i are colored for all i, so v_k is colored; this is a contradiction.

Problem 7 [15 points] **Algorithms**

Let $G = (V, E)$ be a simple, undirected graph. The following algorithm, *RedBlue*, manipulates partial colorings of G using $C = \{\text{red}, \text{blue}\}$. (Partial colorings are defined in the previous problem.)

> Initially, one vertex r of G is colored red, and all the other vertices are uncolored. At each step, one of two events occurs:
>
> a) Some uncolored vertex v that is adjacent to a red vertex becomes blue.
>
> b) Some uncolored vertex v that is adjacent to a blue vertex becomes red.
>
> If neither rule can be applied, the algorithm terminates.

(a) [3 points] Formalize this algorithm as a state machine; that is, define Q, Q_0, and δ.

Solution. A state q consists of a (total) function f from V to $\{red, blue, uncolored\}$. A start state is any state in which f maps exactly one $v \in V$ to red, and all others to uncolored. The transitions are all of the form:

$$
\begin{aligned}
(q, q') \in \delta \quad \Longleftrightarrow \quad & \exists v \in V \ q.f(v) = \text{uncolored} \\
& \wedge \big[(q'.f(v) = \text{red} \wedge \exists w \ f(w) = \text{blue} \wedge (v, w) \in E) \\
& \quad \vee (q'.f(v) = \text{blue} \wedge \exists w \ q.f(w) = \text{red} \wedge (v, w) \in E) \big] \\
& \wedge \forall w \neq v \ q.f(w) = q'.f(w)
\end{aligned}
$$

(b) [3 points] Prove that the algorithm eventually terminates.

Solution. We describe a termination function. Define the value of the termination function to be the number d of uncolored vertices. Each step decreases d, so we can appeal to the Termination Theorem and conclude that the algorithm always terminates. Alternatively, we can expand the proof a bit more: since d starts at $n-1$ and is always ≥ 0, it must eventually reach some minimum value (by well-ordering), whereupon the algorithm terminates.

(c) [3 points] Using the fact stated in Problem 6, prove that if G is connected, then all vertices are colored when the algorithm terminates.

Solution. Say the algorithm terminates with some vertex uncolored. Since the graph is connected, there is a path from v to some colored vertex w. By the given fact, there must be an edge along this path for which one endpoint x is colored and the other y is uncolored. We can apply one of the two steps above, assigning to y the opposite of x's color. Thus, the algorithm should not have terminated yet.

(d) [3 points] Consider applying this algorithm to *bipartite* graphs, i.e., graphs in which the vertices can be partitioned into two sets A and B such that there is no edge between any pair of vertices in A, and similarly for B. Suppose the node r that is colored red in the initial state is in set A. Using the Invariant Theorem, prove that in any reachable state, only vertices of A are colored red and only vertices of B are colored blue.

Solution. The Invariant Theorem requires that we show some condition is true in all the start states, and that it is preserved by every step.

$P(n) =$ "After n vertices are colored, all red vertices are in A and all blue vertices are in B."

Base case. In any start state, there is exactly one red vertex in R—by construction—and no blue vertices. Therefore, $P(1)$.

Induction. Assume $P(n)$. Then, consider an execution in which $n+1$ vertices are colored. Consider the vertex i which was colored last. If we uncolor it, we can apply the induction hypothesis and conclude that all red vertices were in R and all blue vertices were in B.

Now we must prove that vertex i is colored correctly. Assume WLOG that vertex i is in B. For i to have been colored, it must have been the case that i was adjacent to a colored vertex; but all vertices adjacent to i are in R (by the definition of bipartite), so i must have been colored blue. This satisfies the invariant theorem, so $P(n+1)$.

Thus, $\forall n \; P(n)$, so all red vertices are in R and all blue vertices are in B.

(e) [3 points] Combine the previous parts to prove that when run on a connected bipartite graph, the above algorithm terminates and outputs a valid 2-coloring.

Solution. We proved the algorithm terminates on connected graphs with all vertices colored. By the invariant above, this means all vertices in R are red and all vertices in B are blue. By the definition of bipartite, there exists no pair of like-colored vertices connected by an edge, so this is a valid 2-coloring.