Quiz 1 Solutions

Problem 1 [10 points] **Quantifiers**

Consider the following predicates where the universe of discourse is the set of all people:

$P(x)$**:** x is a hacker

$Q(x)$**:** x is an MIT student

$R(x)$**:** x is an artist

$S(x)$**:** x is willing to waltz

Express each of the following statements using quantifiers, logical connectives, and and the above predicates:

(a) [1 point] No hackers are willing to waltz.

Solution. $\nexists x \;\; P(x) \wedge S(x)$

(b) [1 point] No artists are unwilling to waltz.

Solution. $\nexists x \;\; R(x) \wedge \neg S(x)$

(c) [1 point] All MIT students are hackers.

Solution. $\forall x \;\; Q(x) \Rightarrow P(x)$

(d) [1 point] MIT students are not artists.

Solution. $\forall x \;\; Q(x) \Rightarrow \neg R(x)$

(e) [4 points] Does (d) follow logically from (a), (b), and (c)? If so, give a proof. If not, give a counterexample.

Solution. Assume (a), (b), and (c). Then, we want to prove that, for any x such that $Q(x)$ (for any MIT student), it follows that $\neg R(x)$ (that student is not an artist).

From (c), we know that $P(x)$ (that the student is a hacker). If we rearrange (a) to

$$\forall x \;\; P(x) \Rightarrow \neg S(x)$$

then clearly $\neg S(x)$ (the student is not willing to waltz). So, if we rearrange (b) to

$$\forall x \;\; \neg S(x) \Rightarrow \neg R(x)$$

it follows that $\neg R(x)$ (the student is not an artist).

(f) [1 point] Translate into English: $(\forall x)(R(x) \vee S(x) \implies Q(x))$.

Solution. All who are artists or people willing to waltz are MIT students.

(g) [1 point] Translate into English: $(\exists x)(R(x) \wedge \neg Q(x)) \implies (\forall x)(P(x) \implies S(x))$.

Solution. If there is an artist who is not an MIT student, then all hackers are willing to waltz.

Problem 2 [10 points] **Induction**

Prove **by induction** (other methods will not receive credit) that

$$(1 \cdot 2) + (2 \cdot 3) + (3 \cdot 4) + \cdots + (n-1)n = \frac{1}{3}(n-1) \cdot n \cdot (n+1)$$

whenever n is a natural number greater than 1.

Solution. Prove $P(n) ::= (1 \cdot 2) + (2 \cdot 3) + (3 \cdot 4) + \cdots + (n-1)n = \frac{1}{3}(n-1) \cdot n \cdot (n+1)$

Basis. $(P(2))$ $(1 \cdot 2) = \frac{1}{3}(1) \cdot 2 \cdot (3)$.

Induction. Assume $P(n)$. Prove $P(n+1)$.

$$\begin{aligned}
\underbrace{(1 \cdot 2) + (2 \cdot 3) + (3 \cdot 4) + \cdots + (n-1)n}_{\text{Apply IH}} + n(n+1) &= \frac{1}{3}(n-1) \cdot n \cdot (n+1) + n(n+1) \\
&= n(n+1)\left[\frac{1}{3}(n-1) + 1\right] \\
&= n(n+1)\left[\frac{1}{3}n + \frac{2}{3}\right] \\
&= \frac{1}{3}n(n+1)\left[n+2\right]
\end{aligned}$$

Problem 3 [10 points] **Structural Induction**

Consider the sets $S_1, S_2, \ldots$ defined inductively as

- $S_0 = \{0, 1\}$
- For all $n \geq 1$, $S_n = \left\{\frac{x+y}{2} \mid x, y \in S_{n-1}\right\}$

i.e., where S_n contains the average values of each pair of numbers in S_{n-1}. Note we allow $x = y$ in the definition of S_n.

(a) [1 point] Write out the elements of each of the following sets: S_1, S_2, and S_3.

Solution.

S_1	$\{0, 1/2, 1\}$
S_2	$\{0, 1/4, 1/2, 3/4, 1\}$
S_3	$\{0, 1/8, 1/4, 3/8, 1/2, 5/8, 3/4, 7/8, 1\}$

(b) [2 points] Generalize (in your own words or in symbols) to give the elements of S_n.

Solution. S_n contains $2^n + 1$ elements (including 0 and 1) evenly spaced along the unit interval; i.e., $S_n = \{i/2^n | 0 \leq i \leq 2^n\}$.

(c) [7 points] Prove by induction that for all $n \in \mathbb{N}$, $1/3 \notin S_n$. You will probably want to strengthen the induction hypothesis. You may not assume your generalization from the previous part is true; you must prove that statement if you wish to use it.

Solution. Strengthen the theorem to

$$P(n) = \text{“For all } a/b \in S_n \text{ with } a, b \in \mathbb{N} \text{ where } a/b \text{ is in reduced form, } b = 2^k \text{ for some } k\text{”}$$

Clearly, if $\forall n\ P(n)$, then $\forall n\ 1/3 \notin S_n$. Thus, we need only prove $\forall n\ P(n)$.

Base cases. $0 = 0/2^0$, $1 = 1/2^0$.

Induction. Assume $P(n)$. By the construction of S_{n+1}, all elements of S_{n+1} are of the form $(x + y)/2$ for some $x, y \in S_n$. Fix any two such elements $x, y \in S_n$, where (by the induction hypothesis) $x = a/2^k$ and $y = c/2^\ell$ with $a, b, k, \ell \in \mathbb{N}$.

Assume WLOG $k \geq \ell$. Then,

$$\begin{aligned}(x + y)/2 &= (a/2^k + c/2^\ell)/2 \\ &= (a/2^k + c2^{k-\ell}/2^k)/2 \\ &= (a + c2^{k-\ell})/2^{k+1}\end{aligned}$$

While this may not be in reduced form, reduction cannot introduce any non-2 factors into the denominator, so this fits the form specified by the theorem. Therefore, $P(n + 1)$.

By induction, $\forall n\ P(n)$. Thus, $\forall n\ 1/3 \notin S_n$.

Problem 4 [10 points] Relations

(a) [1 point] What is the common name for the reflexive closure of the “less than” relation?

Solution. less than or equal to

(b) [1 point] What is the common name for the transitive closure of the “child of” relation?

Solution. descendant of

(c) [2 points] What is the common name for the inverse of the “teaches” relation (that is, if xTy means x teaches y, how do we say $aT^{-1}b$)?

Solution. is taught by

(d) [3 points] Consider the partial order "aRb iff a divides b without remainder" on the universe of natural numbers **greater than 1**. What are the minimal elements under this relation usually called?

Solution. primes

(e) [3 points] Is the relation "xRy iff $x = y$ or x is **not** y's roommate" an equivalence relation? Why or why not?

Solution. No, because it is not transitive. If x and y are roommates but neither is a roommate of z, then xRz and zRy but $\neg xRy$.

Problem 5 [10 points] **Precedence Relations**

Death is planning the end of the world. This involves a number of tasks each of which takes one minute to complete. The prerequisites associated with these tasks are listed below.

Abbrv.	Task	Prerequisites
C	Compose a requiem	
N	Notify the UN	B
D	Signal the daemons	B
B	Blow the trumpet	
T	Sell T-shirts: "All I got was this lousy t-shirt."	N
Q	Grade the 6.042 quiz	S
G	Open the gates	D,N
S	Put out the sun	C,N

(a) [5 points] Represent the tasks and their prerequisites as a directed graph.

Solution. This is very similar to the constructions in the notes.

(b) [5 points] Death is omnipotent and can therefore work on as many tasks at a time as he wishes. What is the minimum amount of time required for him to end the world? Why? (An explanation without proof is fine.)

Solution. 4 time units, since that is the length of the critical path.

Problem 6 [5 points] **Graphs**

Let $G = (V, E)$ be a simple, undirected graph, and let C be a set of colors. Define a *partial coloring of G using C* to be a function that assigns to each node in V either a color in C or no color. (That is, it colors some, none, or all of the nodes using colors in C.) Define a *partly colored graph* to be a simple, undirected graph $G = (V, E)$ together with a partial coloring of G.

Prove the following fact about partly colored graphs:

> In a partly colored graph, any walk connecting a colored node to an uncolored node must include a colored node adjacent to an uncolored node.

Solution. Consider the path $p = v_1 v_2 \cdots v_k$, of which v_1 is colored and v_k is uncolored. Assume there is no edge (v_i, v_{i+1}) such that v_i is colored and v_{i+1} is uncolored. Then, by induction starting from v_1, v_i are colored for all i, so v_k is colored; this is a contradiction.

Problem 7 [15 points] **Algorithms**

Let $G = (V, E)$ be a simple, undirected graph. The following algorithm, *RedBlue*, manipulates partial colorings of G using $C = \{\text{red}, \text{blue}\}$. (Partial colorings are defined in the previous problem.)

> Initially, one vertex r of G is colored red, and all the other vertices are uncolored. At each step, one of two events occurs:
>
> a) Some uncolored vertex v that is adjacent to a red vertex becomes blue.
>
> b) Some uncolored vertex v that is adjacent to a blue vertex becomes red.
>
> If neither rule can be applied, the algorithm terminates.

(a) [3 points] Formalize this algorithm as a state machine; that is, define Q, Q_0, and δ.

Solution. A state q consists of a (total) function f from V to $\{\mathit{red}, \mathit{blue}, \mathit{uncolored}\}$. A start state is any state in which f maps exactly one $v \in V$ to red, and all others to uncolored. The transitions are all of the form:

$$\begin{aligned}(q, q') \in \delta \quad \Longleftrightarrow \quad & \exists v \in V \ q.f(v) = \text{uncolored} \\ & \wedge \big[(q'.f(v) = \text{red} \wedge \exists w \ f(w) = \text{blue} \wedge (v, w) \in E) \\ & \quad \vee (q'.f(v) = \text{blue} \wedge \exists w \ q.f(w) = \text{red} \wedge (v, w) \in E) \big] \\ & \wedge \forall w \neq v \ q.f(w) = q'.f(w)\end{aligned}$$

(b) [3 points] Prove that the algorithm eventually terminates.

Solution. We describe a termination function. Define the value of the termination function to be the number d of uncolored vertices. Each step decreases d, so we can appeal to the Termination Theorem and conclude that the algorithm always terminates. Alternatively, we can expand the proof a bit more: since d starts at $n-1$ and is always ≥ 0, it must eventually reach some minimum value (by well-ordering), whereupon the algorithm terminates.

(c) [3 points] Using the fact stated in Problem 6, prove that if G is connected, then all vertices are colored when the algorithm terminates.

Solution. Say the algorithm terminates with some vertex uncolored. Since the graph is connected, there is a path from v to some colored vertex w. By the given fact, there must be an edge along this path for which one endpoint x is colored and the other y is uncolored. We can apply one of the two steps above, assigning to y the opposite of x's color. Thus, the algorithm should not have terminated yet.

(d) [3 points] Consider applying this algorithm to *bipartite* graphs, i.e., graphs in which the vertices can be partitioned into two sets A and B such that there is no edge between any pair of vertices in A, and similarly for B. Suppose the node r that is colored red in the initial state is in set A. Using the Invariant Theorem, prove that in any reachable state, only vertices of A are colored red and only vertices of B are colored blue.

Solution. The Invariant Theorem requires that we show some condition is true in all the start states, and that it is preserved by every step.

$P(n) =$"After n vertices are colored, all red vertices are in A and all blue vertices are in B."

Base case. In any start state, there is exactly one red vertex in R—by construction—and no blue vertices. Therefore, $P(1)$.

Induction. Assume $P(n)$. Then, consider an execution in which $n+1$ vertices are colored. Consider the vertex i which was colored last. If we uncolor it, we can apply the induction hypothesis and conclude that all red vertices were in R and all blue vertices were in B.

Now we must prove that vertex i is colored correctly. Assume WLOG that vertex i is in B. For i to have been colored, it must have been the case that i was adjacent to a colored vertex; but all vertices adjacent to i are in R (by the definition of bipartite), so i must have been colored blue. This satisfies the invariant theorem, so $P(n+1)$.

Thus, $\forall n\ P(n)$, so all red vertices are in R and all blue vertices are in B.

(e) [3 points] Combine the previous parts to prove that when run on a connected bipartite graph, the above algorithm terminates and outputs a valid 2-coloring.

Solution. We proved the algorithm terminates on connected graphs with all vertices colored. By the invariant above, this means all vertices in R are red and all vertices in B are blue. By the definition of bipartite, there exists no pair of like-colored vertices connected by an edge, so this is a valid 2-coloring.