Problem Set 10.5

Due Date: Never.

Self-study:
Handout 45 on Expectation and Variance.
Self-study problems: Roulette example in Section 2.1 and Gore example in Section 4.3.

Reading:
Lecture Notes 23 and 24 and Handout 46 on Wald's Theorem.

Problems

Problem 1 Eight men and seven women, all single, happen randomly to have purchased single seats in the same 15-seat row of a theater.

(a) What is the probability that the first two seats contain a marriageable couple, i.e., a single man next to a single woman?

(b) What is the expected number of pairs of adjacent seats which contain marriageable couples? (For example, if the sequence of men and women in the seats is

MMMWMMMMWWWWWWM,

then there are four possible couples sitting in the pairs of adjacent seats that start at the 3rd, 4th, 7th and 14th seats, respectively.)

Problem 2 One hundred twenty students take the 6.042 final exam. The mean on the exam is 90 and the lowest score was 30. You have no other information about the students and the exam, e.g., you should not assume that the final is worth 100 points.

(a) State the best possible upper bound on the number of students who scored at least 180.

(b) Give an example set of scores which achieve your bound.

(c) If the maximum score on the exam was 100, give the best possible upper bound on the number of students who scored *at most* 50.

Problem 3 A couple plans to have children until they have a boy. What is the expected number of children that they have, and what is the variance?

Problem 4 Suppose that n people have their hats returned at random. Let $X_i = 1$ if the ith person gets his or her own hat back and 0 otherwise. Let $S_n = \sum_{i=1}^{n} X_i$, so S_n is the total number of people who get their own hat back. Show that

(a) $\mathrm{E}\left[X_i^2\right] = 1/n$.

(b) $\mathrm{E}\left[X_i X_j\right] = 1/n(n-1)$ for $i \neq j$.

(c) $\mathrm{E}\left[S_n^2\right] = 2$. *Hint*: Use (a) and (b).

(d) $\mathrm{Var}\left[S_n\right] = 1$.

(e) $\Pr(S_n \geq 11) \leq .01$ for any $n \geq 11$. *Hint*: Use Chebyshev's Inequality.

Problem 5 Let X and Y be independent random variables taking on integer values in the range 1 to n uniformly. Compute the following quantities:

(a) $\mathrm{Var}\left[aX + bY\right]$

(b) $\mathrm{E}\left[\max(X, Y)\right]$

(c) $\mathrm{E}\left[\min(X, Y)\right]$

(d) $\mathrm{E}\left[|X - Y|\right]$

(e) $\mathrm{Var}\left[|X - Y|\right]$.

Problem 6 Suppose you are playing the game "Hearts" with three of your friends. In Hearts, all the cards are dealt to the players, in this case the four of you will each have 13 cards.

(a) What is the expectation and variance of the number of hearts in your hand?

(b) What is the expectation and variance of the number of suits in your hand?

Problem 7 We have two coins: one is a fair coin and the other is a coin that produces heads with probability 3/4. One of the two coins is picked, and this coin is tossed n times.

(a) Does the Weak Law of Large Numbers allow us to *predict* what limit, if any, is approached by the expected proportion of heads that turn up as n approaches infinity? Briefly explain.

(b) How many tosses suffice to make us 95% confident which coin was chosen? Explain.

Problem 8

(a) Let X be a random variable whose value is an observation drawn uniformly at random from the $\{i \in \mathbb{Z} \mid -n \leq i \leq n\}$. Let $Y = X^2$. Show that $\mathrm{E}[XY] = \mathrm{E}[X]\,\mathrm{E}[Y]$. Are X and Y independent?

(b) Show that in general for any two random variables X and Y that if $\mathrm{E}[XY] = \mathrm{E}[X]\,\mathrm{E}[Y]$ then $\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$.

Problem 9 Consider the following 3-stage experiment:

In the first stage, I throw a 6-sided die. The number of spots that I get determines how many instances of the second stage to perform.

In the second stage, I throw a 6-sided die. This stage is repeated once for each spot I got in the first stage. The total number of spots that I get in this stage determines how many instances of the third stage to perform.

In the third stage, I throw a 6-sided die. This stage is repeated once for each spot I got in all instances of the second stage.

For example, if the outcome of the first stage is 3, then stage 2 gets played 3 times. Suppose the outcomes of the 3 instances of stage 2 are $1, 3, 4$. Then stage 3 is played $1 + 3 + 4 = 8$ times, and the score is the sum of 8 instances of stage 3.

Assume that all throws are mutually independent.

My score is the total number of spots on all the instances of the third stage.

What is my expected score?

(Analyze this as carefully as you can, using Wald's theorem. Be careful about uses of independence.)

Problem 10 Suppose that someone is infected with an unknown incurable contagious disease. Everyday he encounters n healthy people and infects each of them with probability p. Encounters are mutually independent. These people in turn meet n people and infect them with probability p and so on.

(a) Write a recurrence for the expected number of people on the $k+1$st day in terms of the expected number of people on the kth day. Assume that each day every sick person meets n *different* persons.

(b) Solve the recurrence of part (a).

(c) Now suppose that any sick person recovers from the disease each day with probability r. Find the expected number of sick people on the kth day.

Problem 11 In this problem, we analyze probabilistically the first stage of bubble sort performed on a list of n integers $\{i_1, \ldots i_n\}$. We assume that all the integers are distinct.

On the first step we compare integers i_1 and i_2. If $i_1 > i_2$ we swap them otherwise we leave them as they are.

On step k $(2 \leq k \leq n-1)$, we compare the new i_k with i_{k+1}. If $i_k > i_{k+1}$ we swap else we leave them as they are.

We assume that the order of the integers is chosen uniformly from all possible orderings.

(a) If a number x is randomly (uniformly) chosen from a set of n distinct numbers, what is the probability that x is the largest number in the set?

What is the probability that on the kth step $(2 \leq k \leq n-1)$, i_{k+1} is the biggest integer to appear in the list so far?

(b) Let C_k be the indicator random variable for the event "$i_{k+1} > i_k$". What is $\mathrm{E}\left[C_k\right]$?

(c) Let random variable $C = \sum_{k=1}^{n} C_k$. Give an asymptotic bound for $\mathrm{E}\left[C\right]$ as well as an interpretation of what $\mathrm{E}\left[C\right]$ is.