Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
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Handout 30
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Problem Set 7.5

Due Date: Never.

Problems:

Problem 1 How many ways are there to distribute eight balls into six distinct boxes with the first two boxes having collectively at most four balls if

(a) the balls are identical?

(b) the balls are distinct?

Problem 2 Solve the following counting problems.

(a) Suppose that Math Moose starts at position $(0, 0)$. At any time, he can step north (incrementing the first coordinate) or step east (incrementing the second coordinate). In how many different ways can he reach position (n, n), where n is a non-negative integer?

(b) How many different ways are there to list the numbers $1, 2, \ldots, 2n$ such that the even numbers appear in increasing order and the odd numbers appear in decreasing order?

(c) Describe a bijection that maps each one of Math Moose's paths to a listing of the numbers $1, 2, \ldots 2n$ with even numbers increasing and odd numbers decreasing.

Problem 3 A pizza house is having a promotional sale. Their commercial reads:
"... buy 2 large pizzas at regular price, get up to 11 different toppings for each pizza absolutely free. That's 4194304 different ways to design your order!!!! ..."

(a) Show that $4194304 = \left(\sum_{k=0}^{11} \binom{11}{k}\right)^2$ is actually wrong and give the right answer for the number of ways that you can choose toppings for the two pizzas.

(b) Now assume that you can choose any number of the same topping as long as the total number of toppings is no more than 11. (That is, you can have 2 meatball, 4 ham and 5 sausage toppings if you are truly a meat lover.) How many different ways can you now choose toppings for your two pizzas?

Problem 4 A computer operating system requires file names to be exactly four letters long; uppercase and lowercase letters are distinct.

(a) How many possible file names are there?

(b) The operating system is changed so that uppercase and lowercase are no longer distinct. Figure out the new number of file names in two ways: applying the division rule to **(a)** and directly.

(c) The operating system is changed so that file names may be *up to* four letters long. Repeat **(a)** and **(b)**. Can the division rule still be used?

Problem 5

(a) How many ways can $2n$ people be divided into n pairs?

(b) How many ways can you choose n out of $2n$ objects, given that n of the $2n$ objects are identical?

Problem 6 Find the coefficient of

(a) x^5 in $(1+x)^{11}$;

(b) a^2b^8 in $(a+b)^{10}$;

(c) a^6b^6 in $(a^2+b^3)^5$;

(d) x^3 in $(3+4x)^6$.

(e) x^{10} in $(x+(1/x))^{100}$.

(f) x^8y^9 in $(3x+2y)^{17}$.

(g) x^k in $(x^2-(1/x))^{100}$.

Problem 7 We wish to prove the identity

$$\sum_{r=0}^{n} \binom{n}{r}^2 = \binom{2n}{n},$$

for $n \in \mathbb{N}$.

(a) Use the binomial theorem and the fact that

$$(1+x)^n(1+x^{-1})^n = x^{-n}(1+x)^{2n}$$

to prove the identity.

(b) Consider a set of $2n$ objects, of which n are red and n are blue. Use the fact that

$$\binom{n}{r}^2 = \binom{n}{r}\binom{n}{n-r}$$

to give a combinatorial proof of the identity.

Problem 8 Prove that

$$\sum_{a=0}^{n}\sum_{b=0}^{n-a}\frac{n!}{a!b!(n-a-b)!} = 3^n .$$

[Hint: find a formula for $(x+y+z)^n$ and then set $x=y=z=1$.]

Problem 9 Prove or disprove the following statements about sets. (We use "collection" as a synonym for "set" to make some of the statements easier to read.)

(a) The product of a finite collection of countable sets is countable.

(b) The product of a countable collection of finite sets is countable.

(c) The set of all finite subsets of a countable set is countable.

(d) Suppose that the intersection of a countable collection of sets is finite. Then there exists a finite subcollection of the sets with finite intersection.

Problem 10 Use a diagonalization argument to show that the set of functions from the positive integers to the set $\{0,1,2,3,4,5,6,7,8,9\}$ is uncountable.

Problem 11 A real number is *algebraic* if it is the root of a polynomial with rational number coefficients. Algebraic numbers form an important part in the study of (what else) modern algebra. A number that is not algebraic is called *transcendental*.

(a) Let $S_1, S_2, S_3, \ldots$ be a collection of countably many sets. Suppose each S_i is countable. Prove $\cup S_i$ is countable.

(b) Prove that for any particular d, the set of degree d polynomials with rational coefficients is countable, for all $d \geq 1$.

(c) Prove that the set of all roots of degree d polynomials with rational coefficients is countable (you may use the fact that a degree d polynomial has at most d roots).

(d) Prove that the set of algebraic numbers (which is just the union of the sets in the previous part) is countable.

(e) Prove that there is a trancendental number.

Problem 12 In this problem we will prove the remarkable fact that there exist mathematical functions that computers, no matter how powerful, simply cannot compute. We will do this through countability arguments.

(a) Show that the set of finite length binary strings is countable.

(b) From your answer to the previous part, what can you conclude about the countability of the set of all computer programs?

(c) Show that the set of *infinite* length binary strings is uncountable.

(d) A function is a *decision function* if it maps finite length bit strings into the range $\{0, 1\}$. Let F be the set consisting of *all possible* decision functions. Show that set F is uncountable.

(e) A function is computable if there is a computer program that computes it. From your answers to the previous parts, prove that there is a decision function that is not computable.