Practice Questions for Final

Problem 1. Short Answers.

(a) An assembly line produced 10001 cars which passed inspection and 19 which did not. What is the largest n for which there must have been a run of n consecutive cars that passed inspection? _____________ **Solution:** 501 **endsoln**

(b) A bridge hand consists of 13 cards. The *suit-distribution* of a hand is the number of cards in each suit. For example, one possible suit-distribution is 4 spades, 3 hearts, 3 diamonds, and 3 clubs. How many suit-distributions are there?

Solution: This is equivalent to placing 13 balls (corresponding to each card in a hand) to 4 bins (corresponding to each suit). Thus the number of distributions is $\binom{13+4-1}{4-1} = 560$. **endsoln**

(c) In an employment interview for an expedition to the Sahara desert, a weather forecaster honestly promises at least 95% accuracy for his daily predictions of sunshine versus rain. Is this an impressive credential for the job? Briefly explain.

Solution: Unimpressive: he could be 95% accurate by always predicting sunshine. It would be impressive if his daily prediction also was 95% accurate *given that it rained.* **endsoln**

(d) In a simple path in an undirected tree with n vertices, what is the largest number of edges possible? __________

Solution: $n-1$ **endsoln**

(e) What is the coefficient of x^{12} in the polynomial $(7x+3)^{50}$? (Your answer may contain binomial coefficients and integer powers.)

Solution: $3^{50-12}7^{12}\binom{50}{12}$ **endsoln**

(f) Write a closed form expression for

$$\prod_{k=1}^{\infty} 2^{1/2^k}. \quad \underline{\hspace{3cm}}$$

Solution: $2^{\sum_{k=1}^{\infty} 1/2^k} = 2^1 = 2$. **endsoln**

Problem 2. Pigeon Hole Principle

(a) Six people communicate via radio frequency with one another. Each pair of people choose one of **two** available frequencies with which to communicate. Show that there is a subset of 3 people who are each using the identical frequency with which to communicate with others in the subset.

Solution: Each person communicates with five other persons. By the pigeonhole principle, this person (say a) communicates with another three persons (say b,c,and d) using the same frequency. If either of the pairs $(b, c), (b, d), (c, d)$ uses this frequency then we have three persons using it to communicate with each other and we are done. If not, then the pairs $(b, c), (b, d), (c, d)$ all use the other frequency, so we are done again. **endsoln**

(b) Now suppose there are seventeen people communicating via radio frequency with one another. Each pair of people choose one of **three** available frequencies with which to communicate. Show that there is a subset of 3 people who are each using the identical frequency with which to communicate with others in the subset.

Solution: Now each person communicates with sixteen other people, and by pigeonhole uses the same frequency with at least six of them. If any pair among these six persons uses that same frequency then we are done. If not, then these six persons communicate with one another using two different frequencies so we are back in part (a) of the problem. **endsoln**

Problem 3. True/False?

In this problem, A, B and C denote arbitrary sets. f and g are functions as specified in each part and I_A denotes the identity function on the set A. For each statement indicate whether it is **T**rue or **F**alse. No explanation is required.

True:There exists an injection (one-to-one) $f : A \to B$ if and only if there exists a surjection (onto) $g : B \to A$.

False:Let $f : A \to B$ and $g : B \to A$. $f \circ g = I_B$ implies $g \circ f = I_A$.

True:Let $f : A \to B$ and $g : B \to C$. If both f and g are injections, then $g \circ f$ is an injection.

False:Let $f : A \to B$ and $g : B \to C$. $g \circ f$ is a surjection implies both f and g are surjections.

False:Let $f : A \to B$ be an injection. If $|B| \leq |A|$, then f is a bijection.

True:Adding k edges to an undirected graph increases its chromatic number (the number of colors needed to color the vertices) by at most k.

False:The number of length-3 paths in an n-vertex undirected graph is $O(n^2)$.

False:The number of length-3 paths in an n vertex tree is $O(n)$

True:Reversing the directions of all edges in a directed graph leaves the number of simple directed cycles unchanged.

False:If an undirected graph has the degree of every vertex divisible by a prime p, then p must evenly divide the number of edges.

Problem 4. Cardinality. Let $P(A)$ denote the powerset of a set A. Prove there is no bijection between S and $P(P(S))$ for any set S.

Solution: By contradiction. If there was such a bijection, then there is an surjection from S to $P(P(S))$. But there is an obvious surjection $P(P(S))$ to $P(S)$, namely map every one-element set $\{T\} \in P(P(S))$ to $T \in P(S)$ (and map all the other elements of $P(P(S))$ to an arbitrary member of $P(S)$—say the empty set). Since the composition of surjections is a surjection, we conclude there is an surjection from S to $P(S)$, contradicting Cantor's Theorem. **endsoln**

Problem 5. Inclusion-exclusion. Consider the seven letter word *almeyer*.

(a) [5 pts] How many different sequences of these seven letters are there? ____________

Solution: $\binom{7}{1,1,1,2,1,1} = 7!/2!$ **endsoln**

(b) [10 pts] How many such sequences are there that do not contain either of the words *eye* or *ram*? ____________

Solution:

$$\begin{aligned}
\text{Arrangements with eye: use } \{eye, a, l, m, r\} &= 5! \\
\text{Arrangements with ram: use } \{ram, l, e, y, e\} &= 5!/2! \\
\text{Arrangements with ram and eye: use } \{ram, eye, l\} &= 3! \\
\text{Arrangements with neither ram nor eye} &= 7!/2! - 5! - 5!/2! + 3! = 2346.
\end{aligned}$$

endsoln

Problem 6. Bridge

A bridge hand consists of 13 cards. The *suit-distribution* of a hand is the number of cards in each suit. For example, one possible suit-distribution is 4 spades, 3 hearts, 3 diamonds, and 3 clubs.

(a) How many different suit-distributions are there?

Solution: This is equivalent to placing 13 balls (corresponding to each card in a hand) to 4 bins (corresponding to each suit). Thus the number of distributions is

$$\mathrm{BP}(13, 4) = \binom{13+4-1}{4-1} = 560.$$

endsoln

(b) How many different suit-distributions are there assuming that you get at least one card in each suit?

Solution: By using the same correspondence as in part (a), the number of distributions is

$$\mathrm{BP1}(13, 4) = \binom{13-1}{4-1} = 220.$$

endsoln

Problem 7. Conditional Probability. We consider a hypothetical Institute with only two departments: EECS and Math, and consider the experiment where we pick a random applicant for admission to one of these departments. Define the following events:

$$\begin{aligned} \text{A} &::= \text{applicant is accepted,} \\ \text{F} &::= \text{applicant is female,} \\ \text{M} &::= \text{applicant is male,} \\ \text{MATH} &::= \text{applicant applied to Math,} \\ \text{EECS} &::= \text{applicant applied to EECS.} \end{aligned}$$

Assume that all applicants are either male or female, and that no applicant applied to both departments.

(a) The Institute is sued for gender discrimination, with the plaintiff observing that in both EECS and Math, the probability that a woman is accepted is less than the probability that a man is accepted. This observation can be precisely expressed with two inequalities between conditional probabilities involving the events above. State these inequalities.

Solution:

$$\begin{aligned} \Pr(A \mid F \cap EECS) &< \Pr(A \mid M \cap EECS), \\ \Pr(A \mid F \cap MATH) &< \Pr(A \mid M \cap MATH). \end{aligned}$$

endsoln

(b) The Institute's defense attorneys retort that overall among all applicants, a woman is *more* likely to be accepted than a man. Express this observation as an inequality between conditional probabilities involving the events above.

Solution:

$$\Pr\{A \mid F\} > \Pr\{A \mid M\}$$

endsoln

(c) Although the assertions above may appear to be logically inconsistent, both the plaintiff's and the defense's claims about the conditional probabilities can be correct. (Never mind whose interpretation of the data we find more persuasive.) Fill in the missing table entries below with possible applicant statistics for which both the plaintiff and the University attorneys' assertions are true.

	Applied	Accepted	percent accepted
Dept	Females	Females	Females
Math			
EECS	100	70	70%
Total			
Dept	Males	Males	Males
Math	100	50	50%
EECS			
Total			

Solution:

Math	0 females accepted, 1 applied	0%
	50 males accepted, 100 applied	50%
EECS	70 females accepted, 100 applied	70%
	1 male accepted, 1 applied	100%
Overall	70 females accepted, 101 applied	$\approx 70\%$
	51 males accepted, 101 applied	$\approx 51\%$

endsoln

Problem 8. Roulette wheel

You are given a roulette wheel which contains the numbers from 1 to 2n arranged in some way. The wheel has a spinner and on each spin, you get the two numbers at the opposite ends of the spinner. How would you arrange the numbers on the wheel to maximize the expected value of:

1. The sum of the numbers chosen?

   Solution: It makes no difference how we arrange the numbers. The expected value is the same. Why? If we label the ends of the spinner we can define 2 random variables X and Y. X is the number pointed to by one end of the spinner, and Y is the other number. We wish to maximize $E(X+Y)$. But clearly $E(X) = E(Y) = (2n+1)/2$ and by linearity of expectation, $E(X+Y)$ is just $E(X) + E(Y) = 2n+1$. **endsoln**

2. The product of the numbers chosen?

   Solution: The best way to do this is to ensure that 1 is opposite 2, 3 is opposite 4, $\ldots$, and $2n-1$ is opposite $2n$. Let's call this arrangement A^*. This will maximize the expected value of the product.

   Consider any arrangement that isn't the one described above. It therefore must contain an odd i that isn't paired up with $i+1$. Pick the smallest such i and call it k. Say k is opposite p and $k+1$ is opposite q. We know that p and q are bigger than k (by choosing k to be the smallest such number we are assured that all numbers smaller than k are "properly" paired up). Now consider the new arrangement where k is paired with $k+1$ and p is paired with q. It has a better expected product than the previous arrangement.

To prove this note that since all pairs are equally likely when we turn the spinner, it suffices to show that the sum of the products of the numbers (denoted by S) is larger in the second arrangement.

$$\begin{aligned} S_2 - S_1 &= k(k+1) + pq - kp - (k+1)q \\ &= k^2 + k + pq - kp - kq - q \\ &= k^2 + (1-p-q)k + (p-1)q \\ &= (k-q)(k-(p-1)) \end{aligned}$$

Thus $S_2 - S_1$ is greater than zero if k is smaller than both q and $p-1$. But this is true because of our choice of k. Thus the second arrangement is better.

Now we can transform any arrangement that isn't equal to A^* by repeatedly applying the following operation:

"Find the smallest odd i that isn't paired up with $i+1$. If i is paired with p and $i+1$ with q then pair i with $i+1$ and p with q"

Each time we perform this operation, we increase the expected value of the product. Thus A^* must be the best arrangement since it is the only arrangement where we cannot perform this operation. **endsoln**

Problem 9. Expectation and Variance

Suppose the entries a, b, c, d of a 2×2 matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

are chosen independently at random according to the uniform distribution on $\{0, 1, 2, 3, 4\}$. Let $X = \det(M) = ad - bc$.

(a) What is $E(X)$?

Using linearity of expectation

$$E(X) = E(ad - bc) = E(ad) - E(bc) = 0$$

because ad and bc are identically distributed.

(b) What is $E(X \mid b = 1)$?

Using linearity of expectation and the rule $E(XY) = E(X)E(Y)$ for independent random variables we have

$$\begin{aligned} E(X|b=1) &= E(ad|b=1) - E(bc|b=1) \\ &= E(ad) - E(c) \\ &= E(a)E(d) - E(c) \\ &= 2 \cdot 2 - 2 = 2 \end{aligned}$$

(c) What is $Var(X)$?

$$\begin{aligned}
Var(X) &= E(X^2) - E(X)^2 \\
&= E(X^2) \\
&= E((ad-bc)^2) \\
&= E(a^2d^2 + b^2c^2 - 2abcd) \\
&= E(a^2)E(d^2) + E(b^2)E(c^2) - 2E(a)E(b)E(c)E(d) \\
&= 6\cdot 6 + 6 \cdot 6 - 2\cdot 2^4 = 40
\end{aligned}$$

Problem 10. Probability of Graph Assertions. A digraph with vertices $\{1, 2, \ldots, n\}$ is called an n-*digraph.* (The edges of an n-digraph may be *any* pair of vertices; self-loops are allowed.) Given an n-digraph, G, let "$d(x, y)$" be the assertion that G has a directed edge from vertex x to vertex y.

(a) [1 pts] How many n-digraphs are there? ____________ **Solution:** 2^{n^2} **endsoln**

(b) [1 pts] How many n-digraphs satisfy the assertion $\forall x, y : d(x, y)$? ____________**Solution:** 1 **endsoln**

(c) [3 pts] How many n-digraphs satisfy the assertion $\exists x, y : d(x, y)$? ____________**Solution:** $2^{n^2} - 1$ **endsoln**

Suppose we randomly select an n-digraph, with all n-digraphs equally likely to be selected.

(d) [5 pts] Explain why there is nonzero probability that the selected n-digraph satisfies *both* of the following assertions:

$$\forall x_1 \exists x_2 \exists x_3 \forall x_4 \forall x_5 \exists x_6 : d(x_1, x_2) \vee d(x_3, x_4) \vee d(x_5, x_6) \vee d(x_3, x_1)$$
$$\exists x_1 \forall x_2 \forall x_3 \exists x_4 \exists x_5 \exists x_6 : d(x_1, x_2) \wedge d(x_3, x_4) \wedge d(x_5, x_6) \wedge d(x_3, x_1)$$

Solution: We need only show that some n-digraph satisfies both assertions. The complete n-digraph is one. **endsoln**

(e) A directed Hamiltonian cycle in a digraph is a directed cycle going through each vertex exactly once. What is the expected number of directed Hamiltonian cycles in the selected graph?

Solution:

$$\begin{aligned}
&\mathrm{E}\left[\text{number of cycles through a particular Hamiltonian cycle of vertices}\right] \\
&\quad = \Pr\left\{\text{all edges present in a particular cycle of the vertices}\right\} \\
&\quad = 1/2^n.
\end{aligned}$$

The number of Hamiltonian cycles is $n!/n$ so the expected number of cycles is $n!/n2^n$. **endsoln**

Problem 11. Various Games

What is your expected return in each of the following games? You may assume the various coin tosses and dice rolls are independent.

(a) You get to roll six ordinary six-sided dice. You win the number of dollars equal to the total of the numbers came up on the dice rolls.

Solution: The expected return from a single dice is $7/2$. By linearity of expectation the total expected return is $6(7/2) = 21$. **endsoln**

(b) You roll one ordinary die. You win a number of dollars that is equal to six times the number that came up on the die.

Solution: Again by linearity of expectation, the expected return is $6(7/2) = 21$. **endsoln**

(c) You toss a fair coin six times. For each time it comes up heads, you toss an ordinary six-sided die. You win the number of dollars equal to the total number of spots that appear on all the dice.

Solution: If you get to roll i times, the expected win is, by linearity of expectation, $i(7/2)$. So the expected win is

$$\begin{aligned}
&\sum_{i=1}^{6} (\text{expected win in } i \text{ rolls}) \cdot \Pr\{i \text{ rolls}\} \\
&= \sum_{i=1}^{6} (7/2)i \cdot \Pr\{i \text{ rolls}\} \\
&= (7/2)\mathrm{E}[\text{number of heads}] \\
&= (7/2)3 = \frac{21}{2}.
\end{aligned}$$

endsoln

(d) You toss a fair coin repeatedly, until it comes up "tails" for the first time. (This could involve infinitely many tosses, if "tails" never comes up.) After each coin toss, you toss an ordinary six-sided die. You win the number of dollars equal to the total number of spots that appear on all the dice.

Solution:

By the same reasoning as in the previous part, the expected win is

$$\begin{aligned}
&\sum_{i=1}^{\infty} (\text{expected win in } i \text{ rolls}) \cdot \Pr\{i \text{ rolls}\} \\
&= (7/2) \sum_{i=1}^{\infty} i \Pr\{i \text{ rolls}\} \\
&= (7/2)\mathrm{E}[\text{number of rolls}].
\end{aligned}$$

We know the expected number of rolls is the same as the expected number of tosses, which is $1/(1/2) = 2$, so the expected win is 7.
endsoln

(e) You roll two six-sided dice. You win the product of the number of spots that appear on the two dice.

Solution: $E[X]^2 = \frac{49}{4}$ **endsoln**

(f) You roll one ordinary die. You win a number of dollars that is the square of the number of spots.

Solution: $E[X^2] = \frac{1+4+9+16+25+36}{6} = \frac{91}{6}$ **endsoln**

Problem 12. Emotional Quotient.

There is a newly developed test called EQ (*Emotional Quotient*), that is used to decide whether or not you will succeed in life. EQ is expressed as a nonnegative integer. The average EQ over all humans is 1000.

(a) Knowing only the average EQ, find an upper bound for the fraction of humans that have EQ≥ 1500?

Solution: We use Markov's inequality $\Pr(X \geq cE[X]) \leq 1/c$, with $c = 1.5$, to obtain the upper bound $2/3$. **endsoln**

(b) Assuming that the standard deviation of EQ is 100, what fraction of humans can have EQ≥ 1500?

Solution: We use Chebychev's inequality $\Pr(X \geq E[X] + c\sigma_X) \leq \frac{1}{1+c^2}$ with $c = 5$ to get an upper bound of $\frac{1}{26}$ **endsoln**

Problem 13. Spider's dinner

The spider is expecting guests and wants to catch 500 flies for her dinner. The expected number of flies caught in the web per hour is 40.

(a) What is an upper bound to the probability that the spider will have caught 500 flies in 10 hours?

Solution: Using Markov's inequality we obtain $\Pr(X = (5/4)400) \leq \Pr(X \geq (5/4)400) \leq 4/5$ **endsoln**

(b) What is a lower bound to the probability that the spider will have to wait more than 10 hours to catch 50 flies?

Solution: The event that the fly has to wait more than 10 hours to catch 500 flies is the same as the event that the spider will catch less than 500 flies in 10 hours. This is the complement of the event that the spider catches at least 500 flies in 10 hours, so by part (a) the answer is $1 - 4/5 = 1/5$. **endsoln**

(c) The spider checks her log and discovers that the standard deviation for the number of flies caught in an hour is 20. Find an upper bound on the probability that the spider will catch 500 flies within 10 hours, using this additional information.

Solution: Since the standard deviation for the number of flies caught in one hour is 20, the standard deviation for the number of flies caught in 10 hours is $10 \cdot 20$, by the following analysis:

$$\begin{aligned}\mathrm{Var}[10X] &= \mathrm{E}[(10X)^2] - (\mathrm{E}[10X])^2 \\ &= \mathrm{E}[100X^2] - (10\mathrm{E}[X])^2 \\ &= 100\mathrm{E}[X^2] - 100(\mathrm{E}[X])^2 \\ &= 100(\mathrm{E}[X^2] - (\mathrm{E}[X])^2) \\ &= 100\mathrm{Var}[X]\end{aligned}$$

so the standard deviation is $\sqrt{100\mathrm{Var}[X]} = 10\sqrt{\mathrm{Var}[X]}$, or ten times the standard deviation of the number of flies caught in one hour.
Using Chebychev's inequality we obtain

$$\Pr(X = 500) \leq \Pr(X \geq 400 + \frac{1}{2} \cdot 200) \leq \frac{1}{1 + \left(\frac{1}{2}\right)^2} = \frac{4}{5}$$

endsoln