Problems for Recitation 5

Problem 1. We begin with two large glasses. The first glass contains a pint of water, and the second contains a pint of wine. We pour 1/3 of a pint from the first glass into the second, stir up the wine/water mixture in the second glass, and then pour 1/3 of a pint of the mix back into the first glass. We now repeat this pouring back-and-forth process a total of n times.

(a) Describe a closed form formula for the amount of wine in the first glass after n back-and-forth pourings.

(b) What is the limit of the amount of wine in each glass as n approaches infinity?

Problem 2. **(a)** Give an example of functions $f, g : \mathbb{N} \to \mathbb{N}$ such that neither $f = O(g)$ nor $g = O(f)$. Prove it.

(b) Prove or give a counterexample: If $f_1, f_2, \ldots$ is a sequence of functions from $\mathbb{N}$ to $\mathbb{N}$ such that $f_i(n) = O(1)$ for all $i \geq 1$, then

$$\sum_{i=1}^{n} f_i(n) = O(2^n).$$

Problem 3. **(a)** Show that the Master Theorem does not apply to

$$S(m) = S(m/2) + (\log m)^2.$$

(b) Find a $\Theta()$ bound for S anyway.

(c) Find a $\Theta()$ bound for

$$T(n) = T(\sqrt{n}) + (\lg \lg n)^2.$$

The base condition is $T(2) = 1$.