Massachusetts Institute of Technology | Midterm Exam
6.042J/18.062J: Mathematics for Computer Science | November 4, 1998
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# Midterm Exam

**WRITE YOUR NAME HERE:** ______________________________

- This exam is open book, open notes.
- **Show your work as fully and as clearly as possible**; partial credit will be given.
- Write you solutions in the spaces provided. If you run out of space, use the back of the page.
- Do not discuss this exam with students who have not yet taken it.

| Problem | Points | Score |
|---|---|---|
| 1 | 6 | |
| 2 | 12 | |
| 3 | 12 | |
| 4 | 12 | |
| 5 | 5 | |
| 6 | 8 | |
| 7 | 8 | |
| 8 | 8 | |
| 9 | 5 | |
| 10 | 10 | |
| 11 | 6 | |
| 12 | 8 | |
| TOTAL | 100 | |

**Problem 1.** **(6 points)** Consider the following proposition.

$$\forall x \forall y \; (x < y \rightarrow \exists z \; (x < z \wedge z < y))$$

**(a)** **(2 points)** Prove or disprove this proposition, assuming that the universe is the set of integers.

**(b)** **(4 points)** Write the negation of this proposition. Your solution may *not* use the $\neg$ symbol, but may use any of the relations $=$, $<$, $>$, $\leq$, or $\geq$.

**Problem 2.** **(10 points)** Define the matrix

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Let $M^n$ denote the product of $n$ copies of the matrix $M$. Use ordinary induction to prove that for all $n \geq 1$, the upper right entry of $M^n$ is $F_n$, the $n$-th Fibonacci number. State your inductive hypothesis clearly.

For reference, the standard formula for the product of $2 \times 2$ matrices is given below.

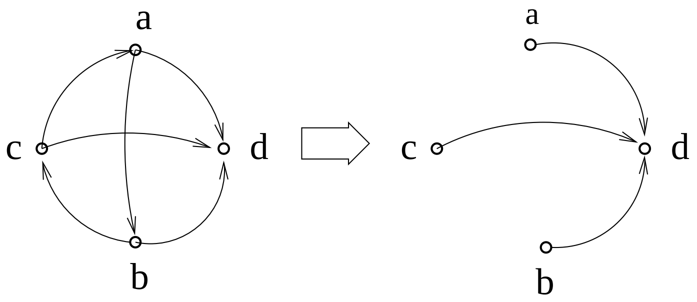
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Recall that $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $\ldots$ .

**Problem 3.** **(12 points)** The vertices of a directed graph can be partitioned into *strongly-connected components*. Two vertices $u$ and $v$ belong to the same strongly connected component if there is a path from $u$ to $v$ and a path from $v$ to $u$. A strongly connected component may consist of a single vertex.

Theory Pig has noticed a strange phenomenon. He starts with a directed graph and partitions the vertices into strongly-connected components. He then removes all edges that connect two vertices in the same strongly-connected component. The resulting graph always seems to be acyclic.

An example is worked out in the figure below. The original directed graph is shown on the left. The vertices $a$, $b$, and $c$ form one strongly-connected component. The single vertex $d$ forms a second strongly-connected component. Theory Pig removes the edges $a \to b$, $b \to c$, and $c \to a$, since these edges connect vertices in the same strongly-connected component. The resulting graph is shown on the right. Note that this graph is acyclic.




Use proof by contradiction to show that for every directed graph, application of Theory Pig's procedure gives a directed acyclic graph.

**Problem 4.** **(12 points)** One of the three monks working on the famed Towers of Hanoi project recently rubbed his pained back and burst out, "Yo! What are we doing? This is for chumps! Let's punt!" But before wandering off to start up fast food joints, they must evenly divide the monastery's collection of prayer beads.

Initially, monk $A$ has 5 beads, monk $B$ has 3 beads, and monk $C$ has 4 beads. The monastic order has strict rules regarding the exchange of prayer beads. Only the following transactions are allowed.

1. Monk $B$ may give a bead to monk $A$ at any time.
2. If $C$ has an odd number of beads, then monk $A$ may give a bead to monk $B$.
3. If $C$ has an even number of beads or monk $A$ has at least two more beads then monk $B$, then monk $C$ may give or take a bead from either monk $A$ or monk $B$.

**(a)** **(3 points)** Model the situation with a state machine. Define the set of states,the set of start states, and the set of transitions.

**(b)** **(3 points)** Prove that the monks can reach the state where monk $A$ has 0 beads, monk $B$ has 9 beads, and monk $C$ has 3 beads by describing a sequence of steps leading to this state.

(c) **(6 points)** Prove that the monks can not reach the state where every monk has 4 beads by using the Invariant Theorem.

**Problem 5.** **(5 points)** Find a closed-form expression equal to the following summation.

$$\sum_{j=1}^{n} \sum_{i=0}^{\infty} \sqrt{j} \cdot \left(1 - \frac{1}{2\sqrt{j}}\right)^{i}$$

**Problem 6. (8 points)** Order the functions listed below so that each function is either $O$ or $\Theta$ of the next function. Indicate whether the stronger $\Theta$ relationship holds in each case. No proofs are required.

- $F_n$, the $n$-th Fibonacci number
- $G(n)$, the number directed graphs with vertices $\{1, 2, 3, \ldots, n\}$ (self-loops permitted)
- $P(n)$, the number of permutations of an $n$-element set
- $M(n)$, the number of $n \times n$ matrices with only 0's and 1's as elements

**Problem 7.** **(8 points)** The inhabitants of Xoros 19 are unsatisfied with the performance of their Fritolator devices. They have constructed a factory to convert Fritolators into new, more deadly craft. Each day, the factory performs one of the following conversions:

$$\begin{aligned} \text{1 Fritolator} &\rightarrow \text{1 Pringlizer} \\ \text{2 Fritolators} &\rightarrow \text{1 Tostitomatic} \\ \text{2 Fritolators} &\rightarrow \text{1 Santitotron} \end{aligned}$$

Grzzwk, the ruler of Xoros 19, plans to use the factory to convert his entire fleet of $n$ Fritolators. However, he must decide the sequence of ships that the factory will produce. Such a sequence is called a *conversion schedule*. For example, there are 5 possible conversion schedules for a fleet of $n = 3$ Fritolators:

1. Pringlizer, Pringlizer, Pringlizer
2. Pringlizer, Tostitomatic
3. Pringlizer, Santitotron
4. Tostitomatic, Pringlizer
5. Santitotron, Pringlizer

**(a)** **(4 points)** Let $x_n$ be the number of possible conversion schedules for a fleet of $n$ Fritolators. Express $x_n$ with a recurrence relation and sufficient base cases.

**(b)** **(4 points)** Find a closed-form expression for $x_n$, the number of conversion schedules for a fleet of $n$ Fritolators.

**Problem 8. (10 points)** Solve the following recurrence problems.

**(a) (5 points)** Find a closed-form solution to the following recurrence equation.

$$\begin{aligned} y_0 &= 0 \\ y_1 &= 1 \\ y_n &= 7y_{n-1} - 12y_{n-2} + 1 \quad (\text{for } n \geq 2) \end{aligned}$$

**(b) (5 points)** Give a $\Theta$ bound on the solution to the following recurrence. For reference, $\ln 7 \approx 1.9$ and $\ln 8 \approx 2.1$.

$$T(n) = 8T\left(\frac{7n}{8}\right) + n\sqrt{n}$$

**Problem 9.** **(5 points)** Suppose that we have functions $f : A \mapsto B$, $g : B \mapsto C$, and $h : C \mapsto A$ such that for all $a \in A$, $h(g(f(a))) = a$, and for all $b \in B$, $f(h(g(b))) = b$. Proofs are not required for any part of this problem.

**(a)** **(1 points)** Which of the functions $f$, $g$, and $h$ must be injective?

**(b)** **(1 points)** Which of the functions $f$, $g$, and $h$ must be surjective?

**(c)** **(2 points)** Circle the relations that must hold.

| | | |
|---|---|---|
| $\lvert B\rvert > \lvert C\rvert$ | $\lvert C\rvert \geq \lvert A\rvert$ | $\lvert B\rvert \leq \lvert C\rvert$ |
| $\lvert A\rvert = \lvert B\rvert$ | $\lvert A\rvert \neq \lvert C\rvert$ | $\lvert A\rvert + \lvert B\rvert \geq \lvert C\rvert$ |

**(d)** **(1 points)** If $g$ is surjective, then which of the functions $f$, $g$, and $h$ must be bijective?

**Problem 10. (10 points)** Indicate whether each of the following sets is finite (F), countably infinite (CI), or uncountably infinite (UI) by circling the right answer. *Note that an even quantity is necessarily finite.*

| | |
|---|---|
| The set of all binary strings with even length. | F CI UI |
| The set of all infinite binary strings with 1's in every even position. (Odd positions may contain either a 0 or a 1.) | F CI UI |
| The set of all infinite binary strings with an even number of 1's. | F CI UI |
| The set of even primes. | F CI UI |
| The set of all even-size sets of even-length binary strings. | F CI UI |

**Problem 11.** **(6 points)** The Pigeonhole Principle can be used to prove each of the following statements. In each case, say what the pigeons are, what the holes are, and what the function is that maps pigeons to holes.

**Example.** If I pick five cards from a deck, then I will have at least two cards from some suit.

Pigeons: The five cards.

Holes: The four suits.

Function: A card is mapped to its suit.

**(a)** **(2 points)** If everyone in an $n$-player tournament wins at least one match, then there must be two players with the same number of wins.

Pigeons:
Holes:
Function:

**(b)** **(2 points)** This afternoon, 19 pigeons flew through 32 holes in a fence in such a way that at least one pigeon flew through each of the holes. Therefore, some pigeon flew through at least two holes.

Pigeons:
Holes:
Function:

**(c)** **(2 points)** In any collection of six different, non-empty subsets of $\{a, b, c, d, e\}$, there exist two sets that intersect.

Pigeons:
Holes:
Function:

**Problem 12.** **(8 points)** You might be interested to know that 42 students in the class have formed a new living group— 042 House— for students who especially enjoy 6.042 and want to talk about it all the time, even when they are at home. For this problem, your solutions may be expressions.

**(a)** **(4 points)** So far at 042 House there have been three meetings, which house members refer to as $a$, $b$, and $c$. (That's the kind of people they are.) The secretary recorded the following attendance information.

- 34 attended meeting $a$
- 32 attended meeting $b$
- 30 attended meeting $c$
- 28 attended meetings $a$ and $b$
- 27 attended meetings $a$ and $c$
- 26 attended meetings $b$ and $c$
- 25 attended meetings $a$, $b$, and $c$

How many house members punted all three meetings?

**(b)** **(2 points)** The 042 House is about to have its first wild party. They have identified 23 distinct preparatory chores, such as working out a good mix of practice problems to entertain the guests. One student must be assigned to each chore, and no student can be assigned to two or more chores. In how many ways can students be assigned to chores?

**(c)** **(2 points)** Theory Pig has decided to visit the living group for 30 days, provided that one student attend to his needs each day. A student may attend to Theory Pig any number of times. In how many ways can the sequence of attendants be chosen?