Lecture Notes

In the last lecture, we saw a variety of techniques for counting elements in a finite set: the Sum Rule, Inclusion-Exclusion, the Product Rule, tree diagrams, etc. In this lecture we introduce a new counting tool called the *Division Rule* and use it to find a formula for the number of r-*combinations* of a set. One common pitfall in counting problems is mixing up r-combinations with r-permutations; try keep track of which is which!

1 The Division Rule

We will state the Division Rule twice, once informally and then again with more precise notation.

Theorem 1.1 (Division Rule) If B is a finite set and $f : A \mapsto B$ maps precisely k items of A to every item of B, then A has k times as many items as B.

For example, suppose A is a set of students, B is a set of recitations, and f defines the assignment of students to recitations. If 15 students are assigned to every recitation, then the Division Rule says that there are 15 times as many students as recitations.

The following two definitions permit a more precise statement of the Division Rule.

Definition If $f : A \mapsto B$ is a function, then $f^{-1}(b) = \{a \in A \mid f(a) = b\}$.

That is, $f^{-1}(b)$ is the set of items in A that are mapped to the item $b \in B$. In the preceding example, $f^{-1}(b)$ is the set of students assigned to recitation b.

This notation can be confusing, since f^{-1} normally denotes the inverse of the function f. With our definition, however, $f^{-1}(b)$ can be a set, not just a single value. For exampe, if f assigns no items in A to some element $b \in B$, then $f^{-1}(b)$ is actually the empty set. In the special case where f is a bijection, $f^{-1}(b)$ is always a single value, and so f^{-1} by our definition is just the ordinary inverse of f.

Definition A function $f : A \mapsto B$ is *k-to-1* if for all $b \in B$, $|f^{-1}(b)| = k$.

For example, if f assigns exactly 15 students to each recitation, then f is 15-to-1. Assuming k is non-zero, a k-to-1 function is always a surjection; every element of the range is mapped to by $k > 0$ elements of the domain.

We can now restate the Division Rule more precisely:

Theorem (Division Rule, restatement) If B is a finite set and $f : A \mapsto B$ is k-to-1, then $|A| = k|B|$.

Proof. Since B is finite, we can let $n = |B|$ and let $B = \{b_1, b_2, \ldots, b_n\}$. Then we have:

$$A = f^{-1}(b_1) \cup f^{-1}(b_2) \cup \ldots \cup f^{-1}(b_n)$$

Equality holds because each side contains the other. The left side contains the right because every set $f^{-1}(b_i)$ is contained in A by the definition of $f^{-1}(b_i)$. The right side contains the left, since every item in a maps to some b_i, and therefore is contained in the set $f^{-1}(b_i)$.

Furthermore, all of the sets $f^{-1}(b_i)$ are disjoint. The proof is by contradiction. Assume for the purpose of contradiction that $a \in f^{-1}(b_i)$ and $a \in f^{-1}(b_j)$ for some $i \neq j$. The first inclusion implies $f(a) = b_i$, but the second inclusion implies that $f(a) = b_j$. This is a contradiction since $f(a)$ denotes a unique item.

Since the sets $f^{-1}(b_i)$ are disjoint, we can compute the cardinality of their union with the Sum Rule. Note that each set $f^{-1}(b_i)$ has size k, since f is k-to-1.

$$\begin{aligned} |A| &= |f^{-1}(b_1)| + |f^{-1}(b_2)| + \ldots + |f^{-1}(b_n)| \\ &= \underbrace{k + k + \ldots + k}_{n=|B| \text{ terms}} \\ &= k|B| \end{aligned}$$

■

1.1 Seating at a Round Table

In how many ways can King Arthur seat n knights at his round table? Two seatings are considered equivalent if one can be obtained from the other by rotation. For example, if Arthur has only four knights, then there are six possibilities as shown in Figure 1. Hereafter, we denote a seating arrangement in text by listing knights in square brackets in clockwise order, starting at an arbitrary point. For example, [1234] and [4123] are equivalent.

Claim 1.2 Arthur can seat n knights at his round table in $(n-1)!$ ways.

In particular, the claim says that for $n = 4$ knights there are $(4-1)! = 3! = 6$ orderings, which is consistent with the example in Figure 1.

Proof. The proof uses the Division Rule. Let A be the set of orderings of n knights in a line. Let B be the set of orderings of n knights in a ring. Define $f : A \mapsto B$ by $f((x_1, x_2, \ldots, x_n)) = [x_1 x_2 \ldots x_n]$.

The function f is n-to-1. In particular:

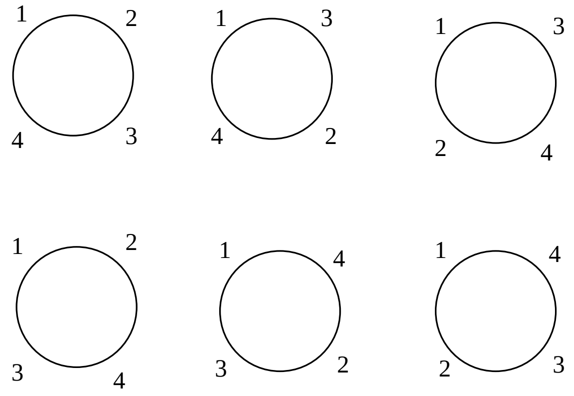

Figure 1: *These are the 6 different ways that King Arthur can seat 4 knights at his round table. Two seatings differing only by rotation are considered equivalent. We denote a seating arrangement in text by listing knights in square brackets in clockwise order, starting at an arbitrary point. For example, the seatings in the top row can be denoted* [1234], [2413], *and* [3421].

$$
\begin{aligned}
f^{-1}([x_1 x_2 \ldots x_n]) \quad = \{ \quad & (x_1, x_2, x_3, \ldots, x_n), \\
& (x_n, x_1, x_2, \ldots, x_{n-1}) \\
& (x_{n-1}, x_n, x_1, \ldots, x_{n-2}), \\
& \ldots x_1 \text{ appears in } n \text{ different places} \ldots \\
& (x_2, x_3, x_4, \ldots, x_1)\}
\end{aligned}
$$

By the Division Rule, $|A| = n|B|$. This gives:

$$
\begin{aligned}
|B| &= \frac{|A|}{n} \\
&= \frac{n!}{n} \\
&= (n-1)!
\end{aligned}
$$

The second equality holds, because the number of orderings of n items in a line is $n!$, as we showed last lecture. ∎

1.2 r-Combinations of a Set

The Division Rule can be used to count the r-combinations of a set, which are defined as follows.

Definition An r-*combination* of a set is a subset of size r.

For example, there are three 2-combinations of the set $\{A, B, C\}$:

$$\{A, B\} \quad \{A, C\} \quad \{B, C\}$$

Understanding the distinction between r-combinations and r-permutations is important. The r-combinations of a set are different from r-permutations in that order does not matter. For example, the above set has six 2-permutations:

$$\begin{array}{ccc} (A, B) & (A, C) & (B, C) \\ (B, A) & (C, A) & (C, B) \end{array}$$

The number of r-combinations of an n-element set comes up in many problems. The quantity is important enough to merit two notations. The first is $C(n, r)$; this is analogous to the notation $P(n, r)$ from last lecture for the number of r permutations of an n-element set. The second notation is $\binom{n}{r}$, read "n choose r". The values $\binom{n}{r}$ or $C(n, r)$ are often called *binomial coefficients* because of their prominent role in the Binomial Theorem, which we will cover shortly.

The following theorem gives a closed form for $\binom{n}{r}$. This theorem is used all the time; remember it!

Theorem 1.3

$$\binom{n}{r} = \frac{n!}{(n-r)!\, r!}$$

The theorem says, for example, that the number of 2-combinations of the three element set $\{A, B, C\}$ is $\frac{3!}{(3-2)!2!} = 3$ as we saw above.

Proof. The proof uses the Division Rule. Let X be a set with n elements. Let A be the set of r-permutations of X, and let B be the set of r-combinations of X. Our goal is to compute $|B|$. To this end, let $f : A \mapsto B$ be the function mapping r-permutations to r-combinations that is defined by:

$$f(\underbrace{(x_1, x_2, \ldots, x_r)}_{\substack{\text{order matters} \\ \text{(permutation)}}}) = \underbrace{\{x_1, x_2, \ldots, x_r\}}_{\substack{\text{order does not matter} \\ \text{(combination)}}}$$

The function f is $r!$-to-1. In particular:

$$\begin{aligned} f^{-1}(\{x_1, x_2, \ldots, x_r\}) = \{ \quad & (x_1, x_2, x_3, x_4, \ldots, x_r), \\ & (x_2, x_1, x_3, x_4, \ldots, x_r), \\ & \ldots \text{all } r! \text{ permutations} \ldots \} \end{aligned}$$

By the Division Rule, $|A| = r!|B|$. Now we can compute the number of r-combinations of an n-element set, $|B|$, as follows:

$$\begin{aligned} |B| &= \frac{|A|}{r!} \\ &= \frac{P(n,r)}{r!} \\ &= \frac{n!}{(n-r)!\,r!} \end{aligned}$$

■

Binomial coefficients are important enough that some special values of $\binom{n}{r}$ are worth remembering:

$$\binom{n}{0} = \frac{n!}{n!\,0!} = 1 \qquad \text{There is a single 0-element subset of an } n\text{-set.}$$

$$\binom{n}{n} = \frac{n!}{0!\,n!} = 1 \qquad \text{This is a single } n\text{-element subset of an } n\text{-set.}$$

$$\binom{n}{1} = \frac{n!}{(n-1)!\,1!} = n \qquad \text{These are } n \text{ 1-element subsets of an } n\text{-set.}$$

Recall that 0! is defined to be 1, not zero.

2 Examples Involving r-Combinations

There is a card game called Five-Card Draw. In this game, each player is dealt a hand consisting of 5 cards from a deck of 52 cards. Each card in the deck has a suit (clubs ♣, hearts ♡, diamonds ♢, or spades ♠) and a value (A, 1, 2, ..., 9, J, Q, or K). The order in which cards are dealt does not matter. We will count various types of hands in Five-Card Draw to show how to use the formula for r-combinations given in Theorem 1.3.

Five-Card Hands

How many different hands are possible in Five-Card Draw? A hand is just a 5-card subset of the 52-card deck. Therefore, the possible hands in Five-Card Draw are exactly the 5-combinations of a 52-element set. By Theorem 1.3, there are $\binom{52}{5} = \frac{52!}{47!\,5!} = 2,598,960$ possible hands.

Confusion between combinations and permutations is a common source of error in counting problems. For example, one might erroneously count the number of 5-card hands as follows. There are 52 choices for the first card, 51 choice for the second card given the first, ..., and 48 choices for the fifth card given the first four. This totals $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$ possible hands. The problem with this way of counting is that a hand is counted once for every ordering of the five cards; that is, each hand is counted $5! = 120$ times. We accidentally counted r-permutations instead of r-combinations! To get the number of 5-combinations, we would have to divide by 120.

Hands with Four-of-a-Kind

How many hands are there with four-of-a-kind? For example, $9\spadesuit\ 4\diamondsuit\ 9\clubsuit\ 9\heartsuit\ 9\diamondsuit$ has a four-of-a-kind, because there are four 9's.

First we must choose one value to appear in all four suits from the set of 13 possible values. This can be done in $\binom{13}{1} = 13$ ways. Then we have to choose one more card from the remaining set of 48. This can be done in $\binom{48}{1} = 48$ ways. In total, there are $13 \cdot 48 = 624$ hands with four-of-a-kind. The situation is illustrated by the tree diagram in Figure 2.

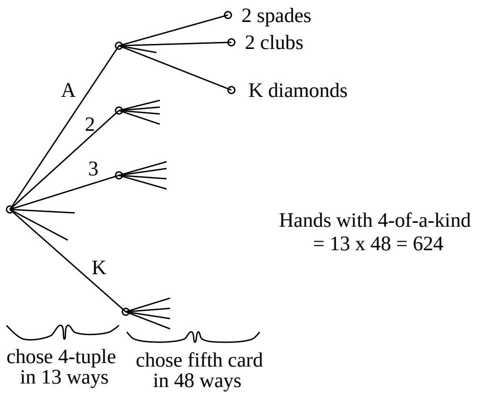

Figure 2: *This (incomplete) tree diagram counts the number of hands with four-of-a-kind. There are 13 choices for the value appearing in all four suits. The fifth card in the hand can be any of the 48 cards remaining in the deck. This gives a total of* $13 \cdot 48 = 624$ *possible hands.*

Here is a way to interpret the result that uses probabilty. If all $\binom{52}{5}$ hands are equally likely, then the probability that we are dealt a four-of-a-kind is $624/\binom{52}{5} = 1/4165$. In other words, we can only expect to get four-of-a-kind only once in every 4165 games of Five-Card Draw! (This is just a passing observation; we will talk about probabilty "for real" in a few weeks.)

Mistakes are easy to make in counting problems. It is a good idea to check a result by counting the same thing in another way. For example, we could also count the number of hands with a four-of-a-kind as follows. First, there are 52 ways to choose the "extra" card. Given this, there are 12 ways to choose the value that appears in all four suits. (We can not choose the value of the first card.) The possibilities are illustrated by the tree diagram in Figure 3. By this method, we find that there are $52 \cdot 12 = 624$ four-of-a-kind hands. This is consistent with our first answer.

Hands with a Full House

A full house is a hand with both a three-of-a-kind and a two-of-a-kind. For example, the hand $7\spadesuit\ 7\diamondsuit\ J\clubsuit\ J\heartsuit\ J\diamondsuit$ is a full house because there are three jacks and two sevens. How many different full houses are there?

We can choose the value that appears three times in 13 ways. Then we can pick any three of the four cards in the deck with this value; this can be done in $\binom{4}{3} = 4$ ways. There are then 12

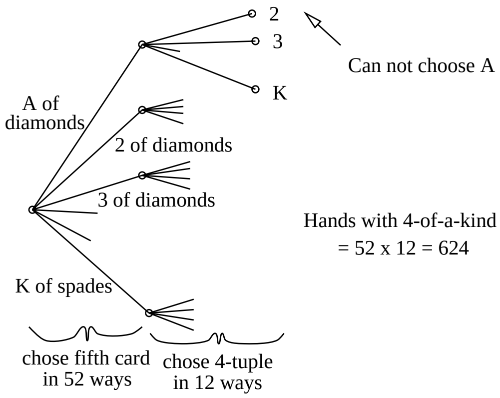

Figure 3: *This tree diagram shows another way to count the number of hands with a four-of-a-kind. There are 52 choices for the "extra" card and then 12 choices for the value appearing in all four suits. This gives* $52 \cdot 12 = 624$ *hands, the same answer as before.*

remaining choices for the value that appears two times. We can pick any two of the four cards with this value; this can be done in $\binom{4}{2} = 6$ ways.

The total number of full house hands is therefore $13 \cdot 4 \cdot 12 \cdot 6 = 3744$. This is 6 times greater than the number of hands with a four-of-a-kind. Since a four-of-a-kind is more rare, it is worth more in poker!

Hands with Two Pairs

How many hands are there with two pairs, but no three- or four-of-a-kind?

Incorrect count. There are 13 choices for the value of the cards in the first pair. There are then $\binom{4}{2} = 6$ ways to choose two of the four cards in the deck that have this value. Next, there are 12 choices for the value of the cards in the second pair, and $\binom{4}{2} = 6$ ways to choose two of the four cards with this value. Finally, the fifth card can be any one of the 48 cards remaining in the deck. Alltogether there are $13 \cdot 6 \cdot 12 \cdot 6 \cdot 48$ hand with just two pairs.

There are actually *two* bugs in this argument. The first is that the fifth card can *not* be any one of the 48 remaining cards; in particular, it can not have the same value as a card already selected. (Otherwise, there would be a three-of-a-kind.) This rules out 4 of the 48 cards, leaving only 44 choices for the fifth card.

The second bug is that every hand has been counted twice! For example, a pair of kings and a pairs of queens is counted once with the kings as the first pair and a second time with the queens as the first pair. The references in the argument to a "first pair" and a "second pair" signal danger; these terms imply an ordering that is not part of the problem. To fix the second bug, we can apply the Division Rule. There is a 2-to-1 mapping from the set of hands we counted to the set of hands with two pairs. Therefore, the set of hands with two pairs is half as large as our initial count:

$$\begin{aligned}\text{hands with two pairs} &= \frac{13 \cdot 6 \cdot 12 \cdot 6 \cdot 44}{2} \\ &= 123,552\end{aligned}$$

Why did this factor of 2 arise in counting pairs, but not in counting full houses? The reason is that pairs are interchangable, but a pair and a triple are not. For example, a pair of 2's and a pair of 9's is the same as a pair of 9's and a pair of 2's. On the other hand, a pair of 2's and a triple of 9's is very different from a pair of 9's and a triple of 2's!

To check our result, we can count the number of hands with two pairs in a completely different way. The number of ways to choose the values for the two pairs is $\binom{13}{2}$. We can choose suits for each pair in $\binom{4}{2} = 6$ ways, and we can choose the remaining card in 44 ways as before. This gives:

$$\binom{13}{2} \cdot 6 \cdot 6 \cdot 44 = 123,552$$

This is the same answer as before. Two pairs turn out to be 33 times more likely than a full house; as one would expect, a full house beats two pairs in poker!

Hands with No Pairs

How many hands are there with no pairs?

First count. There are 52 choices for the first card, 48 choices for the second card given the first, 44 choices for the third given the first two, etc. After each card, the number of remaining choices drops by four, since we can not subsequently pick the card just taken or any of the other three cards with the same value. This gives $52 \cdot 48 \cdot 44 \cdot 40 \cdot 36$ hands with no pair. Is this answer right?

The references to "first card", "second card", ... are a warning signal that we may be inadvertantly counting an ordered set. To be sure, we can check how many times a particular hand is counted. Consider the hand $A\heartsuit$ $K\spadesuit$ $Q\clubsuit$ $J\clubsuit$ $10\heartsuit$. We could have obtained this hand by drawing cards in the order listed, but we could also have drawn them in another order. In fact, we counted this hand once for every possible ordering of the five cards; that is, each hand is counted $5! = 120$ times! We can correct our initial answer by applying the Division Rule. This gives:

$$\begin{aligned}\text{hands with no pairs} &= \frac{52 \cdot 48 \cdot 44 \cdot 40 \cdot 36}{120} \\ &= 1,317,888\end{aligned}$$

Second count. There is another way to count the hands with no pairs that avoids this confusion. Since there are no pairs, no two cards have the same value. This means that we could first choose 5 distinct values from the set of 13 possibilites. Then we could choose one of the four suits for each of the five cards. This gives:

$$\begin{aligned}
\text{hands with no pairs} &= \binom{13}{5} \cdot 4^5 \\
&= \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \cdot 4^5 \\
&= 1,317,888
\end{aligned}$$

This is the same answer as before. Notice that the number of hands with no pairs is more than half of all the hands. This means that in more than half our games of Five-Card Draw, we will not even have one pair!

Hands with Every Suit

How many hands contain one card of every suit? Such a hand must contain two cards from one suit and one card from each of the other three suits. Again, we can count in two ways.

First count. Pick one card from each suit. This can be done in 13^4 ways. Then pick any one of the remaining 48 cards. This gives a total of $13^4 \cdot 48$ hands. However, we have overcounted by two, since the two cards with the same suit could be picked in either order. Accounting for this, there are $\frac{13^4 \cdot 48}{2} = 685,464$ hands.

Second count First, we can pick the suit that contains two cards in four ways. Then we can pick values for these two cards in $\binom{13}{2}$ ways. Finally, we can pick a card from each of the other three suits in 13^3 ways. This gives:

$$\begin{aligned}
\text{hands with all suits} &= 4\binom{13}{2}\binom{13}{1}^3 \\
&= 4 \cdot \frac{13 \cdot 12}{2} \cdot 13 \cdot 13 \cdot 13 \\
&= 685,464
\end{aligned}$$

More than a quarter of all Five-Card Draw hands contain all suits.

Reminders

To make sure that you get the right answer when counting:

1. Look at a sample 5-card hand (or a sample of whatever you are counting) and make sure that it is covered once and only once in your count.
2. Try counting by two different ways and check that you get the same answer.

For example, when we first counted the number of hands with no pairs, we discovered that we had masively overcounted; a sample hand was actually covered 120 times by our count! We corrected this answer by applying the Division Rule, *and* we counted the same thing again in a different way as a double-check.

3 The Magic Trick

Counting has many applications in computer science. For example, we have proved upper and lower bounds on the run time of algorithms, proved that some functions are not computable, etc. Counting also has applications in information theory. Here is a fun one!

There is a magician and an assistant. The assistant goes into the audience with a deck of 52 cards while the magician looks away. Five audience members are asked to select one card each from the deck. The assistant then gathers up the five cards and announces four of them to the magician. The magician thinks for a time and then correctly names the secret, fifth card!

3.1 How the Trick Works

The assistant has somehow communicated the secret card to the magician just by naming the other four cards. In particular, the assistant has two ways to communicate. First, he can announce the four cards in any order. The number of orderings of four cards is $4! = 24$, so this alone is insufficient to identify which of the remaining 48 cards is the secret one. Second, the assistant can choose which four of the five cards to reveal. The amount of information that can be conveyed this way is harder to pin down. The assistant has $\binom{5}{4} = 5$ ways to choose the four cards revealed, but the magician can not determine which of these five possibilities the assistant selected, since he does not know the secret card! Nevertheless, these two forms of communication allow the assistant to covertly reveal the secret card to the magician.

Here is an overview of how the trick works. The secret card has the same suit as the first card announced. Furthermore, the value of the secret card is offset from the value of the first card announced by between 1 and 6. See Figure 4. This offset is communicated by the order of the last three cards.

Here are the details. The audience selects five cards and there are only 4 suits in the deck. Therefore, by the Pigeonhole Principle, the assistant can always pick out two cards with the same suit. One of these will become the secret card, and the other will be the first card that he announces.

Here is how he decides which is which. Note that for any two card numbers, one is at most six clockwise steps away from the other in Figure 4. For example, if the card numbers are 2 and Q, then 2 is three clockwise steps away from Q. The assistant ensures that the value of the secret card is offset between 1 and 6 clockwise steps from the first card he announces.

The offset is communicated by the order that the assistant announces the last three cards. The magician and assistant agree in advance on an order for all 52 cards. For example, they might use:

$$A\clubsuit, 2\clubsuit, \ldots, K\clubsuit, A\diamondsuit, \ldots, K\diamondsuit, A\heartsuit, \ldots, K\heartsuit, A\spadesuit, \ldots, K\spadesuit$$

With this order, one of the last three cards announced is the smallest (S), one is largest (L), and the other is medium (M). The offset is encoded by the order of these three:

$$\begin{array}{lll} SML = 1 & SLM = 2 & MSL = 3 \\ MLS = 4 & LSM = 5 & LMS = 6 \end{array}$$

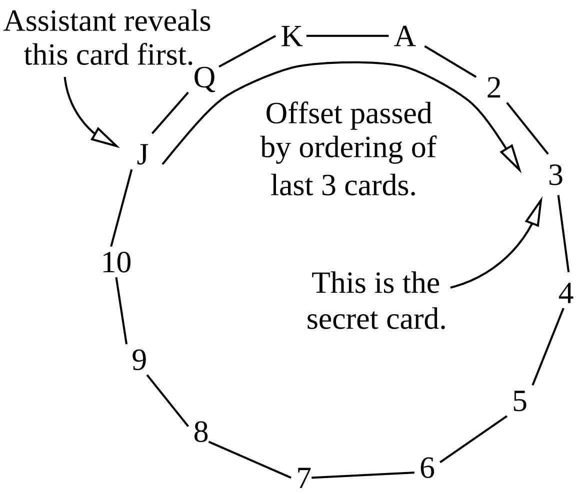

Figure 4: *The 13 possible card values can be ordered in a cycle. For any two distinct values, one is offset between 1 and 6 clockwise steps from the other. In this diagram, 3 is offset five steps from J. In the card trick, the value of the secret card is offset between 1 and 6 steps from the first card announced. This offset is communicated by the order of the last three cards named by the assistant.*

For example, suppose that the audience selects $3\heartsuit$ $8\diamondsuit$ $A\spadesuit$ $J\heartsuit$ $6\diamondsuit$. The assistant picks out two cards with the same suit, say $3\heartsuit$ and $J\heartsuit$. The $3\heartsuit$ is five clockwise steps from $J\heartsuit$ in Figure 4. So the assistant makes $3\heartsuit$ the secret card, announces $J\heartsuit$ first, and then encodes the number 5 by the order in which he announces the last three cards: $A\spadesuit$ $6\diamondsuit$ $8\diamondsuit$ $= LSM = 5$.

3.2 Same Trick with 4 Cards?

Could the same magic trick work with just 4 cards? That is, if the audience picks four cards, and the assistant reveals three, then can the magician determine the fourth card? The answer turns out to be "no". The proof relies on the Pigeonhole Principle.

Theorem 3.1 The magic trick is not possible with 4 cards.

Proof. The audience can select any 4-combination of cards; let A be the set of all such 4-combinations. The assistant can announce any 3-permutation of cards; let B be the set of all such 3-permutations. The formulas for r-combinations and r-permutations give the following sizes for A and B:

$$
\begin{aligned}
|A| &= C(n,r) \\
&= \frac{52!}{48!\,4!} \\
&= 270,725 \\
|B| &= P(n,r) \\
&= \frac{52!}{49!} \\
&= 132,600
\end{aligned}
$$

The assistant sees a 4-combination of cards selected by the audience and must announce a 3-permutation to the magician. Let $f : A \mapsto B$ be the function that the assistant uses in mapping one to the other. Since $|A| > |B|$, the Pigeonhole Principle implies that f maps at least two 4-combinations to the same 3-permutation. That is, there are two different sets of four cards that the audience can pick for which the assistant says exactly the same thing to the magician.

For these two sets of four cards, three cards must be the same (since the assistant announces the same three cards in both cases) and one card must be different (since the two sets are different). For example, these might be the two sets of four cards for which the assistant says exactly the same thing to the magician:

$$3\heartsuit\ 8\diamondsuit\ A\spadesuit\ J\heartsuit$$
$$3\heartsuit\ 8\diamondsuit\ A\spadesuit\ K\diamondsuit$$

In this case, the assistant announces $3\heartsuit$ $8\diamondsuit$ $A\spadesuit$ in some order. The magician is now stuck; he can not determine whether the remaining card is $J\heartsuit$ or $K\diamondsuit$. ∎

There are many variants of the magic trick. For example, if the audience picks 8 cards, then revealing 6 to the magician is enough to let him determine the other two. This sort of subtle transmission of information is important in the security business where one wants to prevent information from leaking out in undetected ways. This is actually a whole field of study, but this trick is all we'll see in 6.042.