Common Errors on Problem Sets

Problem 1. Write the statement "n is prime" in propositional form. Let the universe of discourse be the natural numbers N.

Incorrect answer:

$(n > 1) \wedge (\forall x : (\neg(x = 1) \wedge \neg(x = n)) \wedge (n/x \notin N))$

The problem is in the long chain of ANDs. The last AND should be swapped for an IMPLIES. As it stands, this proposition is satisfied for NO number $n \in N$. One of the things it says is $\forall x : \neg(x = 1)$. There is, however, a value of x for which $(x = 1)$, namely, 1. Students often mistakenly swap $\rightarrow$ and $\wedge$.

The following is a correct answer:

$(n > 1) \wedge (\forall x : (\neg(x = 1) \wedge \neg(x = n)) \rightarrow (n/x \notin N))$

Problem 2. Write the statement "There is a deceitful person who did not give gifts to anyone". Let the universe of discourse be all people. Let $D(x)$ be true iff x is deceitful. Let $G(x, y)$ be true iff x gave a gift to y.

Incorrect answer:

$\exists x \neg \exists y : (D(x) \wedge G(x, y))$

The problem is that the statement does not say that such a person exists. Suppose everyone in the world gave a gift to someone, and at least one person, call her z, is not deceitful. Then the statement "There is a deceitful person who did not give gifts to anyone" is certainly not true. There does exist an x, however, for which there is no person y for which x is deceitful and generous. If $x = z$, then there is no y for which $D(x)$ holds. The problem arises when placing statements, such as $D(x)$ inside a $\exists$ or $\forall$ with which they have nothing to do(i.e. $\exists y : D(x)$.

Another incorrect answer:

$\exists x : (D(x) \rightarrow \neg \exists y : G(x, y))$

The problem here is once again the swapping of $\rightarrow$ and $\wedge$. If $D(x)$ is not satisfied, then you have the case of false implies anything. Once again, if $x = z$, the rest of the statement holds, but there is no deceitful person who was unkind to everyone.

The following is a correct answer:

$\exists x : (D(x) \wedge \neg \exists y : G(x, y))$

Problem 3. A real number r is called *sensible* if there exist positive integers a and b such that $\sqrt{a/b} = r$. For example, setting $a = 2$ and $b = 1$ shows that $\sqrt{2}$ is sensible. Prove that $\sqrt[3]{2}$ is not sensible. (Consider only positive real roots in this problem.)

Incorrect proof:

Proof. The proof is by contradiction. Assume for the purpose of contradiction that $\sqrt[3]{2}$ is sensible. Then there exist positive integers a and b such that $\sqrt{a/b} = \sqrt[3]{2}$. Squaring both sides of this equation gives $a/b = \sqrt[3]{4}$. This can be rewritten as $a = b \cdot \sqrt[3]{4}$. Because $\sqrt[3]{4}$ is not an integer, at least one of a and b is not an integer, which leads to a contradiction. ■

The error is in the last sentence. It states that multiplying an integer by a non-integer yields a non-integer. This is incorrect. Consider the following values for a, b, and c.
$a = 2$, $b = 5$, $c = 2/5$
$(2 = 5 * 2/5) \wedge (2/5 \notin Z)$, but 2 and 5 are both integers.

A correct proof should show that $\sqrt[3]{4}$ is not rational in order to reach a contradiction.

Problem 4. Use induction to prove that the following equation holds for all $n \geq 2$.

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\cdots\left(1 - \frac{1}{n}\right) = \frac{1}{n}$$

Proof. The proof is by induction on n. In the base case, $P(2)$ is true because $(1 - \frac{1}{2}) = \frac{1}{2}$. In the inductive step, for $n \geq 2$ assume $P(n)$ to prove $P(n+1)$ as follows.

$$\begin{aligned}
\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\cdots\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n+1}\right) &= \frac{1}{n+1} \\
P(n) \cdot \left(1 - \frac{1}{n+1}\right) &= \frac{1}{n+1} \\
\frac{1}{n} \cdot \left(1 - \frac{1}{n+1}\right) &= \frac{1}{n+1} \\
\frac{1}{n} \cdot \frac{n}{n+1} &= \frac{1}{n+1} \\
\frac{1}{n+1} &= \frac{1}{n+1}
\end{aligned}$$

This shows that $P(n)$ implies $P(n+1)$ and the claim is proved by induction.

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There are three problems with this proof. The first is that the inductive hypothesis, $P(n)$, was never defined. Even though it may seem obvious to you what it is, you should always define it. The second problem is a misuse of the inductive hypothesis in the proof. The proof makes the statement:

$$P(n) \cdot \left(1 - \frac{1}{n+1}\right) \quad = \quad \frac{1}{n+1}$$

The problem here is that the proof uses $P(n)$ to be the expression

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\cdots\left(1 - \frac{1}{n}\right)$$

In reality, the proposition $P(n)$ is the entire statement of equality, namely

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\cdots\left(1 - \frac{1}{n}\right) \quad = \quad \frac{1}{n}$$

The final problem is the chain of "equalities". The first equality is $P(n+1)$, which isn't known yet to be true. The way to do this correctly is to reverse the equalities to start with something true and work from there.

A corrected proof, by reversing the order of the steps follows:

Proof. The proof is by induction on n. Let $P(n)$ be the proposition that the above equation holds. In the base case, $P(2)$ is true because $(1 - \frac{1}{2}) = \frac{1}{2}$. In the inductive step,for $n \geq 2$ assume $P(n)$ to prove $P(n+1)$ as follows.

$$\begin{aligned}
\frac{1}{n+1} &= \frac{1}{n+1} \\
\frac{1}{n} \cdot \frac{n}{n+1} &= \frac{1}{n+1} \\
\frac{1}{n} \cdot \left(1 - \frac{1}{n+1}\right) &= \frac{1}{n+1} \\
\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\cdots\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n+1}\right) &= \frac{1}{n+1}
\end{aligned}$$

This shows that $P(n)$ implies $P(n+1)$, and the claim is proved by induction.

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