Problem Set 6 Solutions

Problem 1. Find closed-form solutions to the following homogenous linear recurrences.

(a) $x_n = 5x_{n-1} - 6x_{n-2} \quad (x_0 = 0, x_1 = 1)$

The characteristic equation is $r^2 - 5r + 6 = 0$. The roots can be found with the quadratic formula:

$$\begin{aligned} r_1 &= 2 \\ r_2 &= 3 \end{aligned}$$

This implies a solution of the form

$$x_n = A2^n + B3^n.$$

Substituting the initial conditions into this general form gives a system of linear equations:

$$\begin{aligned} 0 &= A + B \\ 1 &= 2A + 3B \end{aligned}$$

The solution to this system is $A = -1$ and $B = 1$. Therefore, the final solution to the recurrence is

$$x_n = 3^n - 2^n.$$

(b) $x_n = 5x_{n-1} - 8x_{n-2} + 4x_{n-3} \quad (x_0 = 1, x_1 = 2, x_2 = 3)$

The characteristic equation is $r^3 - 5r^2 + 8r - 4 = 0$. Solving a cubic equation can be messy process, but in this case the roots are easy to find:

$$\begin{aligned} r_1 &= 2 \\ r_2 &= 2 \\ r_3 &= 1 \end{aligned}$$

The general form of the solution is therefore

$$x_n = A2^n + Bn2^n + C.$$

Substituting the initial conditions into this general form gives a system of linear equations.

$$\begin{aligned} 1 &= A + C \\ 2 &= 2A + 2B + C \\ 3 &= 4A + 8B + C \end{aligned}$$

The solution to this linear system is $A = 2$, $B = -\frac{1}{2}$, and $C = -1$. The complete solution to the recurrence is therefore

$$x_n = 2^{n+1} - n2^{n-1} - 1.$$

Problem 2. Find closed-form solutions to the following inhomogenous linear recurrences.

(a) (Fibonacci Plus) $x_n = x_{n-1} + x_{n-2} + 1 \quad (x_0 = 0, x_1 = 1)$

First, we find the general solution to the homogenous recurrence. The characteristic equation is $r^2 - r - 1 = 0$. The roots of this equation are:

$$\begin{aligned} r_1 &= \frac{1+\sqrt{5}}{2} \\ r_2 &= \frac{1-\sqrt{5}}{2} \end{aligned}$$

Therefore, the solution to the homogenous recurrence is of the form

$$x_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Next, we need a particular solution to the inhomogenous recurrence. Since the inhomogenous term is constant, we guess a linear solution.

$$\begin{aligned} an+b &= a(n-1)+b+a(n-2)+b+1 \\ &= 2an-3a+2b+1 \\ 0 &= an-3a+b+1 \end{aligned}$$

This equation holds if we set $a = 0$ and $b = -1$. Therefore, the inhomogeous recurrence has the simple particular solution $x_n = -1$. The complete solution to the recurrence is the homogenous solution plus the particular solution:

$$x_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n - 1$$

All that remains is to find the constants A and B. Substituting the initial conditions gives a system of linear equations.

$$\begin{aligned} 0 &= A+B-1 \\ 1 &= A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) - 1 \end{aligned}$$

The solution to this linear system is:

$$\begin{aligned} A &= \frac{5+3\sqrt{5}}{10} \\ B &= \frac{5-3\sqrt{5}}{10} \end{aligned}$$

Therefore, the complete solution to the recurrence is

$$x_n = \left(\frac{5+3\sqrt{5}}{10}\right)\cdot\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5-3\sqrt{5}}{10}\right)\cdot B\left(\frac{1-\sqrt{5}}{2}\right)^n - 1.$$

(b) $x_n = 3x_{n-1} - 2x_{n-2} + n \quad (x_0 = 0, x_1 = 1)$

First, we find the general solution to the homogenous recurrence. The characteristic equation is $r^2 - 3r + 2 = 0$. The roots of this equation are $r_1 = 1$ and $r_2 = 2$. Therefore, the general solution to the homogenous recurrence is

$$x_n = A + B2^n.$$

Now we must find a particular solution to the recurrence, ignoring the boundary conditions. Since the inhomogenous term is linear, we guess a solution that is quadratic:

$$\begin{aligned} an^2 + bn + c &= 3(a(n-1)^2 + b(n-1) + c) - 2(a(n-2)^2 + b(n-2) + c) + n \\ &= an^2 + 2an - 5a + bn + b + c + n \\ 0 &= 2an - 5a + b + n \end{aligned}$$

This equation is solved by setting $a = -\frac{1}{2}$ and $b = -\frac{5}{2}$. Apparently, any value of c gives a valid particular solution; for simplicity, we choose $c = 0$. Therefore, the particular solution is:

$$x_n = -\frac{1}{2}n^2 - \frac{5}{2}n$$

The complete solution to the recurrence is the homogenous solution plus the particular solution:

$$x_n = A + B2^n - \frac{1}{2}n^2 - \frac{5}{2}n$$

Substituting the initial conditions gives a system of linear equations:

$$\begin{aligned} 0 &= A + B \\ 1 &= A + 2B - 3 \end{aligned}$$

The solution to this linear system is $A = -4$ and $B = 4$. Therefore, the complete solution to the recurrence is

$$x_n = 2^{n+2} - \frac{1}{2}n^2 - \frac{5}{2}n - 4.$$

Problem 3. The 6.042 staff recently received the following email.

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From: grzzwk@xoros19.gammaquad.plnt
To: 6042-staff@theory.lcs.mit.edu
Subject: Pset 5 issue

Thank you for contributing the MergeSortX19 algorithm to our
''intellectually challenged'' society.  My, my-- we feel mentally
bloated by this sudden infusion of technological wizardry!

As a token of thanks, we launched a Doritofier missile on a straight
line toward Earth.  Suppose that p(t) denotes the distance between
Xoros 19 and the Doritofier after t days of flight.  At the moment of
launch, the Doritofier was here on Xoros 19, and so p(0) = 0.  After a
single day, the Doritofier had already traveled 100 units toward
Earth, so p(1) = 100.  In general, the position of the missle is given
by the following recurrence:

       p(t+1) = 1.99 * p(t) - p(t-1)

Earth happens to be 1010 units from Xoros 19.  Gotta run!  Seeya!

/Grzzwk
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(a) Find a closed-form solution to the recurrence for the distance between Xoros 19 and the Doritofier after t days of flight.

Let $c = 1.99$. The characteristic equation of Grzzwk's recurrence is then $r^2 - cr + 1 = 0$. The roots of this equation are:

$$\begin{aligned} r_1 &= \frac{c + \sqrt{c^2 - 4}}{2} \\ r_2 &= \frac{c - \sqrt{c^2 - 4}}{2} \end{aligned}$$

Therefore, the general solution to the recurrence is:

$$p(t) = A\left(\frac{c + \sqrt{c^2 - 4}}{2}\right)^t + B\left(\frac{c - \sqrt{c^2 - 4}}{2}\right)^t$$

Subtituting the initial conditions into this general form gives a system of linear equations.

$$
\begin{aligned}
0 &= A + B \\
100 &= A\left(\frac{c+\sqrt{c^2-4}}{2}\right) + B\left(\frac{c-\sqrt{c^2-4}}{2}\right)
\end{aligned}
$$

The solution to this linear system is

$$
\begin{aligned}
A &= \frac{100}{\sqrt{c^2-4}} \\
B &= -\frac{100}{\sqrt{c^2-4}}
\end{aligned}
$$

Therefore, the complete solution to the recurrence is

$$
p(t) = \frac{100}{\sqrt{c^2-4}} \cdot \left(\left(\frac{c+\sqrt{c^2-4}}{2}\right)^t - \left(\frac{c-\sqrt{c^2-4}}{2}\right)^t\right)
$$

where $c = 1.99$. Note that $\sqrt{c^2-4}$ is imaginary.

(b) The characteristic equation approach gives a solution to this recurrence involving complex numbers. This makes evalutation awkward. Give a closed-form solution that does not involve complex numbers. DeMoivre's Identity may be helpful:

$$
(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta
$$

We start with the solution obtained previously.

$$
p(t) = \frac{100}{\sqrt{c^2-4}} \cdot \left(\left(\frac{c+\sqrt{c^2-4}}{2}\right)^t - \left(\frac{c-\sqrt{c^2-4}}{2}\right)^t\right)
$$

Note that

$$
\begin{aligned}
\frac{c+\sqrt{c^2-4}}{2} &= \frac{c}{2} + i \cdot \sqrt{1 - \left(\frac{c}{2}\right)^2} \\
&= \cos\theta + i\sin\theta
\end{aligned}
$$

where $\theta = \arccos\frac{c}{2}$. With this substitution, we can eliminate complex numbers from the solution.

$$
\begin{aligned}
p(t) &= \frac{100}{\sqrt{c^2-4}} \cdot \left((\cos\theta + i\sin\theta)^t - (\cos\theta - i\sin\theta)^t\right) \\
&= \frac{100}{\sqrt{c^2-4}} \cdot ((\cos t\theta + i\sin t\theta) - (\cos t\theta - i\sin t\theta)) \\
&= \frac{100}{\sqrt{c^2-4}} \cdot (2i\sin t\theta) \\
&= \frac{100}{\sqrt{1-\left(\frac{c}{2}\right)^2}} \cdot (\sin t\theta) \\
&= \frac{100\sin t\theta}{\sin\theta}
\end{aligned}
$$

The final solution is

$$
p(t) = \frac{100\sin t\theta}{\sin\theta}
$$

where $\theta = \arccos(\frac{1.99}{2})$.

(c) Does the Doritofier missle hit earth?

The solution indicates that the path of the Doritofier is a sine wave with amplitude

$$
\begin{aligned}
\frac{100}{\sin\theta} &= \frac{100}{\sin\arccos\left(\frac{c}{2}\right)} \\
&= \frac{100}{\sqrt{1-\left(\frac{c}{2}\right)^2}} \\
&= \frac{100}{\sqrt{1-\left(\frac{1.99}{2}\right)^2}} \\
&= 1001.253\ldots
\end{aligned}
$$

Since Earth is 1010 units from Xoros 19, we're safe.

On the other hand, since the path of the missle is a sine wave, it *will* be returning to Xorox 19 presently...