Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
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Handout 50
November 10, 1998

Problem Set 9

Due: Start of class on November 17, 1998.

Read sections 1 to 4 in The Essentials of Probability.

Problem 1. **(10 points)** Suppose that you are the contestant on a variant of *Lets Make a Deal* where the keys to a new Corvette are behind one door, tickets for a dream vacation are behind a second door, and there is a diminutive, stuffed pig behind a third door. You do not know which door conceals which prize. Your objective is to maximize your probability of winning the pig.

As before, you pick a door initially. Carol then opens a different door. Specifically, if you picked the door concealing the pig or the keys, then she reveals the tickets. If you picked the door concealing the tickets, then she reveals the keys. You can then either stay with your original door or switch to the remaining, closed door. You are awarded the prize behind the door that you choose at this second stage.

In substance, this variant differs from the original in that you know Carol's door-opening rules. This may give you a slight edge. Give a strategy that maximizes your probability of winning the pig and prove that no better strategy exists.

Problem 2. **(10 points)** In each problem part, either define a probability space and events A, B, and C such that all the equations hold simultaneously, or else prove that no such probability space and events exist.

(a)

$$\begin{aligned}\Pr(A \mid B) &= \frac{1}{4}\\ \Pr(A \mid C) &= \frac{1}{4}\\ \Pr(A \mid B \cap C) &= \frac{3}{4}\end{aligned}$$

(b)

$$\begin{aligned}\Pr(A \mid B) &= \frac{3}{4}\\ \Pr(A \mid C) &= \frac{3}{4}\\ \Pr(A \mid B \cap C) &= \frac{1}{4}\end{aligned}$$

(c)

$$\begin{aligned}\Pr(A \mid B) &= \frac{3}{4}\\ \Pr(A \mid C) &= \frac{3}{4}\\ \Pr(A \mid B \cup C) &= \frac{1}{4}\end{aligned}$$

Problem 3. **(10 points)** Suppose that I pick a pair of distinct numbers in the range 1 to 10 uniformly at random.

(a) If one of the numbers is even, then what is the probability that the other number is even?

(b) If one of the numbers is a 2, then what is the probability that the other number is even?

Problem 4. **(10 points)** Three gunfighters, A, B, and C, plan to fight a three-cornered pistol duel. They all know that A's chance of hitting his target is 0.3, B never misses, and C's chance of hitting is 0.5. They will shoot one at a time in the sequence $ABCABC\ldots$. On his turn, a gunfighter may target either of the other two, or may fire harmlessly into the air. Assume that each gunfighter uses the shooting strategy that maximizes his probability of being the only one left standing. What should gunfighter A do on his first shot and what probability of survival does this give him?

Problem 5. **(10 points)** After their arrest and sentencing for unlawful use of firearms, and short hospital stays, the three gunfighters, A, B, and C applied for parole. The parole board decided to release two of the three, chosen uniformly at random, but has not released their names.

Gunfighter A figures that he will be released with probability $\frac{2}{3}$. A guard offers to tell gunfighter A the name of one of the other gunfighters who will be released (either B or C). However, gunfighter A declines the offer. He reasons that if the guard says, for examle, "B will be released", then his own probability of release will drop to $\frac{1}{2}$. This is because he will then know that either he or C will also be released, but these events are equally likely.

Either prove that gunfighter A has reasoned correctly or prove that he is wrong. Assume that if the guard has a choice of saying B or C because both are to be released, then he picks one uniformly at random.

Problem 6. **(10 points)** Suppose that *Let's Make a Deal* is modified so that there are four doors. As before, a prize is hidden behind one door. The contestant picks a door, Carol opens a different door to reveal no prize, and then the contestant is allowed to stay with his or her original door or to pick one of the other two that are still closed. If the contestant chooses the door concealing the prize in this second stage, then he or she wins it.

(a) Contestant Stu, a sanitation engineer from Trenton, New Jersey, stays with his original door. What is the probability that he wins the prize?

(b) Contestant Zelda, an alien abduction researcher from Helena, Montana, randomly picks one of the other two doors with equal probability. What is the probability that she wins the prize?

Problem 7. **(10 points)** Even if he is very, very careful, the probability that Theory Pig can avoid "provoking" 6.042 TA Brian Dean is only a small, positive constant ϵ. If Theory Pig does "provoke" Brian, then the probability that Brian will refrain from doing something really wicked to our porky pal is only a small, positive constant δ.

(a) Compute a tight lower bound on the probability that Brian does something really wicked to Theory Pig.

(b) Prove that your lower bound is tight; that is, describe circumstances consistent with the problem statement such that the probability that Brian does something wicked to Theory Pig is equal to your lower bound.

Problem 8. **(15 points)** Each morning Theory Pig pops out of his desk drawer home— all bright and perky— and goes bounding off to play around the Laboratory for Computer Science. The desk drawer is located at position $(0, 0)$. At each time step, Theory Pig is equally likely to bound north or east. A northward bound increments the first coordinate of his position, and an eastward bound increments the second coordinate of his position. (Assume that his direction at one time step is independent of his direction at all other time steps.)

At position $(4, 4)$, 6.042 TA Brian Dean is waiting with a bat, for some reason. What is the probability that Theory Pig visits position $(4, 4)$ within his first 10 bounds?

Problem 9. **(15 points)** Suppose that you roll three six-sided dice.

(a) What is the probability that the sum of the first two rolls is 7?

(b) What is the probability that the sum of the first two rolls is 7 and the sum of the last two rolls is 7?

(c) What is the probability that every pair of rolls sums up to 7?

(d) What is the probability that some pair of rolls sums up to 7?