Problem Set 8

Due: Start of class on November 10, 1998.

Read sections 4.3, 4.4, and Appendix 3 in Rosen.

Problem 1. **(15 points)** Solve the following counting problems. Simplifying expressions is unnecessary.

(a) Find the number of 5-card hands with consecutive values, e. g. $8\clubsuit\ 9\heartsuit\ 10\diamondsuit\ J\diamondsuit\ Q\spadesuit$. (Assume that the value of an ace is just above the value of a king.)

(b) Find the number of 5-card hands with exactly three aces.

(c) Find the number of 5-card hands in which every suit appears at most twice.

(d) There are 15 sidewalk squares in front of the Lab for Computer Science. Suppose that Theory Pig can be thrown so that he bounces on 0, 1, 2, or 3 distinct sidewalk squares. How many different throws are possible?

(e) In how many different ways can the numbers shown on a red die, a green die, and a blue die total up to 15? Assume that these are ordinary, 6-sided dice.

(f) How many permutations are there of the letters $MISSISSIPPI$?

(g) Find the number of squences of k non-negative integers with sum equal to n. For example, if $k = 2$ and $n = 3$, then there are four sequences: $0 + 3$, $1 + 2$, $2 + 1$, and $3 + 0$.

(h) Find the number of squences of k positive integers with sum equal to n. For example, if $k = 2$ and $n = 4$, then there are three sequences: $1 + 3$, $2 + 2$, and $3 + 1$.

Problem 2. **(10 points)** Solve the following counting problems.

(a) Suppose that Theory Pig starts at position $(0, 0)$. At any time, he can step north (incrementing the first coordinate) or step east (incrementing the second coordinate). In how many different ways can he reach position (n, n), where n is a non-negative integer?

(b) How many different ways are there to list the numbers $1, 2, \ldots, 2n$ such that the even numbers appear in increasing order and the odd numbers appear in decreasing order?

(c) Describe a bijection that maps each one of Theory Pig's paths to a listing of the numbers $1, 2, \ldots 2n$ with even numbers increasing and odd numbers decreasing.

Problem 3. **(20 points)** Suppose that every two people either know one another or else are strangers. How many people must there be in a room to guarantee that there are either 6 people who all know each other or 6 people who are all strangers to one another?

The solution to this problem is a number called $R(6, 6)$. The solution to the simpler problem involving groups of only 5 people is called $R(5, 5)$. Earlier in the course, we solved a still-simpler version of this problem involving groups of just 3 people in about ten minutes. Finding $R(5, 5)$ and $R(6, 6)$ is a bit more tricky. The following is taken from a well-known text, *Ten Lectures on the Probabilistic Method*, by Joel Spencer:

> Erdos [possibly the 20th century's top mathematician] asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5, 5)$ or they will destroy our planet. In that case, he claims, we should marshall all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6, 6)$. In that case, he believes, we should attempt to destroy the aliens.

Given rumors that a new fleet is forming on Xoros 19, we had better take a stab at $R(6, 6)$. For now, however, we will only try to prove that $R(6, 6) > 17$.

(a) If there are n people in a room, then how many different pairs of people are there?

(b) A *configuration* of a group of people is a setting of who-knows-who. For example, there are two possible configurations of a group of just two people: either they know each other or they are strangers. How many different configurations are possible in a room with n people?

(c) Suppose that there is a specific group of six people who all know each other. How many configurations are possible with this restriction?

(d) In a set of n people, how many different subsets of six people are there?

(e) Use your answers to the preceding two parts to find an upper bound on the number of configurations in which there exists some group of six people who all know each other.

(f) Give an upper bound on the number of configurations in which there exist either six people who all know each other or six people who are all strangers.

(g) Use your answers to parts (b) and (f) to prove that there exists a configuration of 17 people such that no set of six people all know each other or are all strangers. This implies that $R(6, 6) > 17$.

Problem 4. **(10 points)** Find generating functions for the following sequences.

(a) $< 0, 0, 1, 1, 1, 1, 1, \ldots >$

(b) $< 0, 2, 4, 6, 8, 10, \ldots >$

(c) $< 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots >$

(d) $< 2^0, 2^1, 2^2, 2^3, 2^4, 2^5, \ldots >$

(e) $< -1, 0, 1, -1, 0, 1, -1, 0, 1, -1, 0, 1, \ldots >$

Problem 5. **(10 points)** Suppose that $f(x)$ is the generating function for the sequence $a_0, a_1, a_2, \ldots$. Find generating functions for the following sequences.

(a) $< 0, 0, 0, a_0, a_1, a_2, \ldots >$

(b) $< a_0, a_1 + 1, a_2 + 2, a_3 + 3, \ldots >$

(c) $< a_0, 0, a_1, 0, a_2, 0, a_3, 0, \ldots >$

(d) $< a_0, -a_1, a_2, -a_3, a_4, \ldots >$

(e) $< a_0, 0, a_2, 0, a_4, 0, \ldots >$

Problem 6. **(10 points)** For each of the following generating functions, find a closed-form expression for the coefficient of x^n.

(a)

$$\frac{5 - 11x}{4x^2 - 5x + 1}$$

(b)

$$\frac{2x}{(1-x)^3}$$

Problem 7. **(15 points)** In this problem, we use generating functions to compute the number of m bit strings with a multiple of three 1's. For example, if $m = 4$, then there are five such strings:

0000	1110	1101
1011	0111	

(a) Use summation notation and binomial coefficients to describe the number of m-bit strings with a multiple of three 1's.

The following fact will be useful in evaluating this sum. Let $z_1 = 1$, $z_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and $z_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. The expression $\frac{1}{3}(z_1^k + z_2^k + z_3^k)$ is then equal to 1 if k is a multiple of 3 and is equal to 0 otherwise.

(b) Left $f(x)$ be a generating function for the series $< a_0, a_1, a_2, \ldots >$. Give a closed-form expression for the following sum.

$$\sum_{i=0}^{\infty} a_{3i}$$

(c) Recall that $f(x) = (1+x)^m$ is the generating function for binomial coefficients; that is, for $0 \leq k \leq m$, the coefficient of x^k is $\binom{m}{k}$, and for $k > m$, the coefficient of x^k is zero. Combine this fact and the two preceding problem parts to obtain a closed-form expression for the number of m-bit strings with a multiple of three 1's.

(d) Use DeMoivre's formula, restated below, to eliminate complex numbers from your solution.

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Problem 8. **(20 points)** A string of parentheses is called "balanced" if the number of ('s is equal to the number of)'s and the number of)'s never exeeds the number of ('s as the string is read from left to right. For example, (()())((())) is a balanced string, but ())(is not. With three pairs of parentheses, five different balanced strings can be formed. These five strings are listed below.

((()))   (()())   (())()   ()(())   ()()()

In general, the number of balanced strings increases rapidly with the number of pairs of parentheses. Your diligent TA stayed up until dawn, slaving away with a dull pencil stub, working out the following table:[1]

[1] This is a lie; he cranked it out with a C program in about six minutes.

Pairs of Parentheses	Number of Balanced Strings
0	1
1	1
2	2
3	5
4	14
5	42
6	132
7	429
8	1430
9	4862
10	16796
11	58786
12	208012

In an apparently unrelated calculation, your exhausted TA then used his final drops of energy and the last nubbin of his pencil to evaluate $500000 - 1000\sqrt{249999}$ out to 72 decimal places:[2]

$$1.000001000002000005000014000042000132000429001430004862016796058786208012\ldots$$

Compare the table to the irrational number. The goal of this problem is to explain the apparently miraculous correspondence.

(a) Let p_n be the number of balanced strings containing n pairs of parentheses. Express p_n with a recurrence equation and sufficient base cases.

(b) Let $P(x)$ be the generating function for the number of balanced strings; that is,

$$P(x) \quad = \quad p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \ldots .$$

Using the recurrence relation as a guide, find a (non-trivial) equation relating and $P(x)$ and the square of $P(x)$.

(c) Apply the quadratic formula to obtain a closed-form expression for the generating function $P(x)$.

(d) Show that $P(1/1000000) = 500000 - 1000\sqrt{249999}$.

(e) Explain why the digits of this irrational number encode the number of balanced strings of parentheses.

[2]This is also a lie; he plugged the thing into Maple.