# Problem Set 4

**Due: Start of class on October 6, 1988.**

**Problem 1.** **(10 points)** Suppose that the dating game is played as described in class. That is, girls pursue their favorite boys, boys reject all but their top prospect, and the game ends when each boy has a single candidate remaining. Prove or disprove each of the following assertions.

**(a)** There is always some girl who gets her favorite boy.

**(b)** For every number of girls and boys, it is possible for every boy to get stuck with his least favorite girl.

**Problem 2.** **(5 points)** There are two girls, Alice and Betty, and two boys, Steve and Tim. Alice prefers Steve, but the others' preferences are such that every boy-girl dating arrangement is stable. What are Betty, Steve, and Tim's preferences?

**Problem 3.** **(10 points)** Prove or disprove the following claim: for some $n \geq 3$, there exists a set of boys' and girls' preferences such that every dating arrangement is stable.

**Problem 4.** **(15 points)** There are $n$ girls in a tennis tournament. The tournament director hates to see anyone unhappy. Therefore, he plans to adjust the outcome of the tournament according to the following rule: if a player loses all her matches, then he will change all her losses to wins. Of course, changing the results in this way might create a new player who lost all her matches. Therefore, his plan is to continue applying his rule until there is no player who lost all her matches.

The tournament is played and the results are reported to the director. It so happens that the original tournament contains no cycles. The director now begins to adjust the results according to his rule.

**(a)** Use a state machine model and the Invariant Theorem to prove that the tournament director never creates a cycle through repeated application of his rule.

**(b)** Prove that by applying his rule, the director can never make everyone happy. That is, no matter how many times he applies his rule, there is still a player that lost all her matches.

**Problem 5. (15 points)** Suppose that the following procedure is used to multiply two non-negative integers $a$ and $b$.

> Let $x = a$, $y = b$, $p = 0$.
> Repeat the following steps.
>     If $x = 0$, then output $p$ and terminate.
>     Otherwise, if $x$ is even, then let $x = x/2$ and $y = 2y$.
>     Otherwise, if $x$ is odd, then let $x = x - 1$ and $p = p + y$.

**(a)** Model the procedure with a state machine.

**(b)** Formulate an invariant relating $a$, $b$, $x$, $y$, and $p$ that holds at the top of the loop. Your invariant should imply that when $x = 0$, $p = ab$.

**(c)** Prove that the invariant holds.

**(d)** Prove that the procedure terminates.

**Problem 6. (10 points)** Brad Banks, the one 6.042 TA that Theory Pig trusts as his true friend, has promised our mascot a delicious treat. He places Theory Pig on one end of a 1 meter long rubber mat and places the treat at the other end of the mat. Theory Pig then begins to run excitedly toward his treat at the rate of 1 centimeter per second. (Theory Pig is four inches tall and a bit pudgy.)

However, after exactly one second has elapsed, Brad stretches the rubber mat out by 1 meter, so that its total length is now 2 meters. The mat stretches uniformly and instantly, so there are now 2 centimeters of mat behind Theory Pig and 198 centimeters between him and his treat. Theory Pig continues running toward his prize, but whenever another second elapses, Brad stretches the mat out by another meter.

Either prove that Theory Pig never reaches his treat or else compute approximately the number of seconds that he requires.

**Problem 7. (10 points)**

**(a)** Sum the following infinite series.

$$\sum_{i=0}^{\infty} \frac{4^i - 3^i}{5^i}$$

**(b)** For the following summation use the integral method to find upper and lower bounds that differ by at most 0.1. Note that you may need to add the first few terms explicitly and then use integrals to bound the sum of the remaining terms.

$$\sum_{i=1}^{\infty} \frac{1}{i^2}$$

(The actual value of the summation turns out to be $\pi^2/6 = 1.644\ldots$.)

**Problem 8.** **(10 points)** A banker loans \$100,000 to the Addams family so that they can purchase a new dilapidated mansion. At the end of each year for the next 20 years, Morticia Addams will drop by the bank to make a payment on the loan. The banker will immediately invest this returned money at a 7% annual rate of return.

What annual payment should the banker require of the Addams family so that his return after 20 years is the same as if he had invested the whole amount at 7% today instead of making the loan?

**Problem 9.** **(15 points)** The 6.042 staff recently received the following mystifying, but disturbing email.

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From: grzzwk@xoros19.gammaquad.plnt
To: 6042-staff@theory.lcs.mit.edu
Subject: Pset 2 complaint

We found your depiction of our society in problem set 2 simplistic,
inaccurate, and insulting.  We have launched a Fritolator device from
our homeworld at galactic coordinates (0, 0, 0).  During each hour,
the Fritolator can move in any one of the following ways.

  1.  first coordinate +4, second +1, third -1
  2.  first coordinate +1, second -1, third +3
  3.  first coordinate -2, second +3, third +4

Expect Earth, located at galactic coordinates (7, 8, 5), to be
Fritolated promptly.

/Grzzwk
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We are unsure what Fritolation entails, but surmise that it is not good.

**(a)** Model the movement of the Fritolator as a state machine.

**(b)** Either list a sequence of Fritolator moves leading to the destruction of Earth, or use the Invariant Theorem to prove that Earth is safe.