Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
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# Supplement to Lecture 2

## 1 An Induction Proof Format

An induction argument has a regular, but somewhat intricate structure. You can use the following format to ensure that you miss no steps; this format is not a requirement, just something to retreat to if you are a bit confused.

First, define the induction hypothesis $P(n)$ carefully. Then proceed as shown below in the style where an assertion is made first, and then the justification follows in a sequence of indented steps.

$\forall n \geq 0\ P(n)$.

1. (base case) $P(0)$.

   (This is is usually easy, but not always.)

2. (inductive step) $\forall n \geq 0\ (P(n) \rightarrow P(n+1))$

   (a) Fix $n \geq 0$.

   (b) $P(n) \rightarrow P(n+1)$.

      i. Assume $P(n)$.

      ii. $P(n+1)$.
      (This is usually the hard part of an induction proof.)

      iii. Q. E. D. (by implication rule)

   (c) Q. E D. (by definition of $\forall$)

3. Q. E. D. (by induction)

**Example.** We cast a standard induction proof into the above format as a model.

**Theorem 1.1** *For all $n \geq 0$, $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$.*

*Proof.* Let $P(n)$ be the proposition that $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$.

$\forall n \geq 0\ P(n)$.

1. (base case) $P(0)$ holds because $1 = \frac{1(1+1)}{2}$.
2. (inductive step) $\forall n \geq 0\ (P(n) \rightarrow P(n+1))$
   (a) Fix $n \geq 0$.
   (b) $P(n) \rightarrow P(n+1)$.
      i. Assume $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$.
      ii. $1 + 2 + \ldots + n + (n+1) = \frac{(n+1)(n+2)}{2}$
         A. $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$ (by assumption)
         B. $1 + 2 + \ldots + n + (n+1) = \frac{n(n+1)}{2} + (n+1)$ (by adding $n+1$ to each side)
         C. $1 + 2 + \ldots + n + (n+1) = \frac{(n+1)(n+2)}{2}$ (by simplifying)
      iii. Q. E. D. (by implication rule)
   (c) Q. E D. (by definition of $\forall$)
3. Q. E. D. (by induction)

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# 2 Double Induction

Suppose that the function $f(n, k)$ is define for all non-negative integers $n$ and $k$ as follows:

- For all $k \geq 0$, $f(0, k) = 1$.
- For all $n \geq 0$, $f(n, 0) = 1$.
- For all $n, k \geq 0$, $f(n+1, k+1) = f(n, k+1) + f(n+1, k)$.

This description is sufficient to calculate $f(n, k)$ for specific values of $n$ and $k$ as shown in the table below.

| | $k = 0$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $n = 0$ | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 2 | 3 | 4 | 5 |
| 2 | 1 | 3 | 6 | 10 | 15 |
| 3 | 1 | 4 | 10 | 20 | 35 |

The following theorem gives the value of $f(n, k)$ directly.

**Theorem 2.1** *For all $n, k \geq 0$,*

$$f(n,k) \quad = \quad \frac{(n+k)!}{n!\ k!}.$$

For example, the theorem states the $f(3,2) = \frac{5!}{3!\ 2!} = 10$, which matches the value computed in the table above.

There is a difficulty with trying to prove this theorem by induction; namely, there are two variables involved, $n$ and $k$. One solution is to use a *double induction* argument. The overall structure of the argument is an induction on $n$, but to complete the induction step, we use another whole induction argument on $k$.

*Proof.* The proof is by double induction. Let $P(n)$ be the proposition that for all $k \geq 0$, $f(n,k) = \frac{(n+k)!}{n!\ k!}$. In the base case, $P(0)$ is true because, by part one of the definition of $f$, $f(0,k) = 1 = \frac{(0+k)!}{0!\ k!}$. In the inductive step, for $n \geq 0$ assume that for all $k \geq 0$, $f(n,k) = \frac{(n+k)!}{n!\ k!}$ in order to prove that that for all $k \geq 0$, $f(n+1,k) = \frac{(n+1+k)!}{(n+1)!\ k!}$.

We complete the inductive step by using another whole induction argument, this time on $k$. Let $Q(k)$ be the proposition that $f(n+1,k) = \frac{(n+1+k)!}{(n+1)!\ k!}$. In the base case, $Q(0)$ is true because, by part two of the definition of $f$, $f(n+1,0) = 1 = \frac{(n+1+0)!}{(n+1)!\ 0!}$.

In the inductive step, for $k \geq 0$ assume that $f(n+1,k) = \frac{(n+1+k)!}{(n+1)!\ k!}$ in order to prove that $f(n+1,k+1) = \frac{(n+1+k+1)!}{(n+1)!\ (k+1)!}$. We reason as follows:

$$\begin{aligned}
f(n+1,k+1) &= f(n,k+1) + f(n+1,k) \\
&= \frac{(n+k+1)!}{n!\ (k+1)!} + \frac{(n+1+k)!}{(n+1)!\ k!} \\
&= \frac{(n+1)(n+k+1)!}{(n+1)!\ (k+1)!} + \frac{(k+1)(n+1+k)!}{(n+1)!\ (k+1)!} \\
&= \frac{(n+1+k+1)!}{(n+1)!\ (k+1)!}
\end{aligned}$$

The first step uses the third part of the definition of $f$. The second step uses the induction hypotheses from both the induction on $n$ and the induction on $k$. The remaining steps are simplifications. ■