Supplement to Lecture 1

This handout contains two proofs that were presented in Lecture 1. Each proof illustrates a common type of mathematical argument. The first shows that two sets are equal by proving that each contains the other, and the second shows that $\sqrt{2}$ is irrational using a proof by contradiction.

The proofs are presented in a stylized form which may make the structure of the arguments more clear. In this stylized form, an assertion is made first, and then the justification follows in a sequence of indented steps.

1 DeMorgan's Law for Sets

Theorem 1.1 *If A and B are sets, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$.*

This theorem is sometimes called DeMorgan's Law. This name can be confusing, however, since there is an analogous theorem involving boolean variables instead of sets that is also called DeMorgan's Law.

The proof of the theorem uses the standard method for showing that two sets are equal; namely, each set is shown to contain the other.

Proof. Goal: Show that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

1. $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$
   (a) Let x be an element of $\overline{A \cup B}$.
   (b) $x \notin A \cup B$ by definition of complement.
   (c) $x \notin A$ and $x \notin B$ by definition of union.
   (d) $x \in \overline{A}$ and $x \in \overline{B}$ by definition of complement.
   (e) $x \in \overline{A} \cap \overline{B}$ by definition of intersection.
   (f) Q. E. D. by definition of set inclusion.
2. $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$
   (a) Let x be an element of $\overline{A} \cap \overline{B}$.
   (b) $x \in \overline{A}$ and $x \in \overline{B}$ by definition of intersection.
   (c) $x \notin A$ and $x \notin B$ by definition of complement.
   (d) $x \notin A \cup B$ by definition of union.

   (e) $x \in \overline{A \cup B}$ by definition of complement.
   (f) Q. E. D. by definition of set inclusion.

3. Q. E. D. since two-way set inclusion implies equality.

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2 The Square Root of 2 is Irrational

Theorem 2.1 *$\sqrt{2}$ is irrational.*

The following argument is a "proof by contradiction". The idea is to assume that the theorem is false (i.e., $\sqrt{2}$ is rational) and follow the consequences of this assumption until a contradiction is reached. This contradiction implies that the theorem must actually be true (i.e., $\sqrt{2}$ is irrational).

Proof. Goal: Show that $\sqrt{2}$ being rational implies a false conclusion.

1. Assume that $\sqrt{2}$ is rational.

2. False conclusion.

   (a) There exist a and b such that $\sqrt{2} = a/b$ where
      - a is an integer,
      - b is a positive integer, and
      - a and b share no common factor

      by definition of a rational number and by the reducibility of fractions to lowest terms.
   (b) $2b^2 = a^2$ by algebra.
   (c) a is even. (See note below.)
   (d) There exists an integer c such that $a = 2c$ by definition of even.
   (e) $2b^2 = 4c^2$ by substitution.
   (f) $b^2 = 2c^2$ by algebra.
   (g) b is even. (See same note below.)
   (h) a and b share 2 as a common factor.
   (i) Q. E. D since (a) contradicts (h).

3. Q. E. D. by contradiction.

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Note: In two steps indicated above, there are logical jumps that might require some consideration. One could expand the proof to justify these jumps more minutely. Ultimately, however, there is an upper bound on the level of rigor attainable in this course, since we have not defined any axioms. Therefore, our goal is only to write proofs where the reasoning proceeds in small, plausible steps.