Tutorial 5

1 Other Asymptotic Notations

Definition 1 *We say $g(x) = \Omega(f(x))$ if there exists a $c > 0$ and an x_0 such that for all $x \geq x_0$, $g(x) \geq c|f(x)|$.*

As with Big Oh notation, one can also indicate $g(x) = \Omega(f(x))$ by writing $g(x) \geq \Omega(f(x))$ or $g(x) \in \Omega(f(x))$. The following theorem shows that Big Oh and Omega are opposites.

Theorem 1 *$f(x) = O(g(x))$ iff $g(x) = \Omega(f(x))$.*

Proof. By definition of the Big Oh notation, $f(x) = O(g(x))$ iff there exists a $c > 0$ and an x_0 such that for all $x \geq x_0$, $|f(x)| \leq cg(x)$. This is equivalent to the statement that there exists a $c' > 0$ and an x_0 such that for all $x \geq x_0$, $g(x) \geq c'|f(x)|$. (Set c' = 1/c to establish one implication and set c = 1/c' for the other.) By the definition of Omega, the second statement holds iff $g(x) = \Omega(f(x))$. ∎

Here are some examples of the use of Omega: $x^2 = \Omega(x)$, $2^x = \Omega(x^2)$, and $\frac{x}{100} = \Omega(100x + \sqrt{x})$. As a final example, to say that a running time $T(n)$ is asymptotically at least quadratic, one writes $T(n) \geq \Omega(n^2)$. (Remember that this is equivalent to $T(n) = \Omega(n^2)$.)

Sometimes we want to specify a running time precisely up to constant factors. For the example, the running time of an algorithm might be at least quadratic with a small constant and at most quadratic with a big constant. For this, we can use Theta notation.

Definition 2 *$f(x) = \Theta(g(x))$ if $f(x) = O(g(x))$ and $f(x) = \Omega(g(x))$.*

In other words, $f(x) = \Theta(g(x))$ if there exists an x_0 and positive constants c_0 and c_1 such that for all $x \geq x_0$,

$$c_0|g(x)| \leq |f(x)| \leq c_1|g(x)|$$

Here are some examples of Theta notation: $10x^3 - 20x^2 + 1 = \Theta(x^3)$ and $\frac{x}{\ln x} \neq \Theta(x)$. If the running time of an algorithm is $T(n) = \Theta(n^2)$, then $T(n)$ grows quadratically; it is bounded both above and below by functions of the form cn^2.

Definition 3 *$f(x) = o(g(x))$ if for all $c > 0$, there exists an x_0 such that for all $x \geq x_0$, $|f(x)| \leq cg(x)$.*

This definition is a little hard to follow because of all the quantifiers. Intuitively, $f(x) = O(g(x))$ means that g grows at least as fast as f. The new notation $f(x) = o(g(x))$ means that g grows *strictly* faster than f.

Theorem 2 $\frac{x}{\ln x} = o(x)$

Proof. We must show that for all $c > 0$, there exists an x_0 such that for all $x \geq x_0$, we have $|\frac{x}{\ln x}| \leq cx$. If we choose $x_0 = e^{1/c}$, then $|\frac{x}{\ln x}| \leq |\frac{x}{\ln e^{1/c}}| = |cx| = cx$ for all $c > 0$ and $x \geq x_0$. $\blacksquare$

Theorem 3 $\frac{x}{100} \neq o(x)$

Proof. The proof is by contradiction. Assume for the purpose of contradiction that $\frac{x}{100} = o(x)$. Then for all $c > 0$, there exists an x_0 such that for all $x \geq x_0$, $|\frac{x}{100}| \leq cx$. In particular, if $c = 1/200$, there must exist an x_0 such that for all $x \geq x_0$, $|\frac{x}{100}| \leq \frac{x}{200}$. Since this statement is false for all positive x, we have reached a contradiction. $\blacksquare$

The last bit of asymptotic notation is the rarely used Little Omega. It is just the opposite of Little Oh.

Definition 4 $g(x) = \omega(f(x)) \Leftrightarrow f(x) = o(g(x))$

For example, $x^2 = \omega(x)$ since x^2 grows strictly faster than x.

A final note about asymptotic notation. Often one sees expressions like:

$$H_n = \ln n + \gamma + O(\frac{1}{n})$$

We encountered a similar situation with the $\sim$ symbol; the above expression really means that $H_n - \ln n - \gamma = O(\frac{1}{n})$. That is, the error term in the approximation for H_n is at most a constant divided by n.

2 Big-Oh

Recall that a function $f(x)$ is said to be $O(g(x))$ if $\exists c > 0, \exists x_0$ such that $\forall x \geq x_0 : |f(x)| \leq cg(x)$.

One has to be careful when combining inductive proofs with aysmptotic notation. Here is another example of a wrong proof. Recall that F_n denotes the n-th Fibonacci number.

False Theorem 4 $F_n = O(1)$

Proof. The proof is by strong induction. Let $P(n)$ be the predicate that $F_n = O(1)$. In the base case, $P(1)$ is true because $F_1 = 1 = O(1)$. In the inductive step, for $n \geq 1$ assume that $F_1 = O(1), F_2 = O(1), ldots, F_n = O(1)$ to prove that $F_{n+1} = O(1)$. There are two cases. If $n = 1$, then $F_{n+1} = F_2 = 1 = O(1)$. If $n > 1$, then $F_{n+1} = F_n + F_{n-1} = O(1) + O(1) = O(1)$. $\blacksquare$

Where is the bug?! The proof is actually correct; the problem is that it is not a proof of the theorem! What we actually proved is that $\forall n : F_n = O(1)$. It is in fact the case that $\forall n : P(n)$ i.e. $\forall n : F_n = O(1)$ is true. This is because once we fix n, i.e. within the quantifier, F_n is just a specific number such as F_{100} or F_{1000} etc. so it is a constant and hence we can say it is $O(1)$. However, this does not imply the same about the *function* $f(n) = F_n$ which is what we care about.

Next let us compare the asymptotics of two functions. Which is larger *asymptotically*, $x \log x$ or $x^{1.1}$?

Claim 1 $x \ln x = O(x^{1.1})$

Proof.

Let $x_0 = e^{10}, c = 1$. Then $\forall x \geq x_0 \ln x \leq x^{0.1}$. Hence $\forall x \geq x_0, x \ln x \leq x^{1.1}$, i.e., $x \ln x = O(x^{1.1})$.

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Can we make a stronger statement? In this case we can actually say that $x \log x = o(x^{1.1})$. Roughly speaking, $f(x) = o(g(x))$ means that f grows slower than g as x increases. Our definition from class says:

$f(x) = o(g(x))$ if $\forall c > 0, \exists x_0$ such that $\forall x \geq x_0 |f(x)| \leq cg(x)$

Another way to phrase this is to say that $Lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

Claim 2 $x \ln x = o(x^{1.1})$.

Proof. Let us find the limit $Lim_{x\to\infty} \frac{x \ln x}{x^{1.1}}$.

$$= Lim_{x\to\infty} \frac{\ln x}{x^{0.1}}$$

Recall L'Hospital's rule: $Lim_{x\to\infty} \frac{f(x)}{g(x)} = Lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ where $f'(x)$ is the derivative of $f(x)$ w.r.t. x and $g'(x)$ is the derivative of $g(x)$ w.r.t. x. Therefore,

$$Lim_{x\to\infty} \frac{\ln x}{x^{0.1}} = Lim_{x\to\infty} \frac{\frac{1}{x}}{0.1x^{-0.9}}$$

$$= Lim_{x\to\infty} \frac{10}{x^{0.1}} = 0.$$

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3 Can sum 'em

We have seen some methods to find closed forms of sums. Finding closed form solutions is tricky business because it is not always the case that they exist! Here we describe another way to sum called the *perturbation* method. As with all other methods it only works sometimes.

Let $S_n = \sum_{i=0}^{n} a_i$. Then,

$$S_n + a_{n+1} = \sum_{i=0}^{n+1} a_i$$

$$S_n + a_{n+1} = a_0 + \sum_{i=1}^{n+1} a_i$$

$$S_n + a_{n+1} = a_0 + \sum_{i=0}^{n} a_{i+1}$$

How do we use this? Consider the sum from class, $S_n = \sum_{i=0}^{n} ix^i$.

The general approach is to add the $(n+1)$st term to both sides,

$$S_n + (n+1)x^{n+1} = 0 + \sum_{i=0}^{n}(i+1)x^{i+1}$$

$$S_n + (n+1)x^{n+1} = x\sum_{i=0}^{n} ix^i + \sum_{i=0}^{n+1} x^i$$

Then find S_n in the expression on the right, and rename that part,

$$S_n + (n+1)x^{n+1} = xS_n + \frac{1-x^{n+2}}{1-x} - 1$$

Now bring terms involving S_n to one side and solve,

$$(1-x)S_n = \frac{1-x^{n+2}-1+x-(n+1)x^{n+1}+(n+1)x^{n+2}}{(1-x)}$$

$$S_n = \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$$

Let us now go back to the integral method and see how to evaluate the sum

$$\sum_{i=1}^{\infty} \frac{1}{i^2}$$

In a similar fashion to the example in the lecture, this sum, minus its first term, is less than the area under the curve $y = 1/x^2$ from 1 to ∞. That is,

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{1}{i^2} &\leq 1 + \int_1^{\infty} x^{-2}dx \\ &= 2.\end{aligned}$$

Thus, our infinite sum is at most 2, and in particular it converges. In fact, one can prove (using complex analysis) that the exact value of the sum is $\pi^2/6$; we won't prove this here.

Next let's look at the following *doubly infinite* sum

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x^i y^j,$$

where (critically)

$$0 < x, y < 1.$$

We can therefore re-write this as

$$\begin{aligned}
\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x^i y^j &= \sum_{i=0}^{\infty} x^i \left(\sum_{j=0}^{\infty} y^j\right) \\
&= \sum_{i=0}^{\infty} x^i \left(\frac{1}{1-y}\right) \\
&= \left(\frac{1}{1-y}\right) \sum_{i=0}^{\infty} x^i \\
&= \frac{1}{(1-x)(1-y)}.
\end{aligned}$$

4 Can't sum it!

We note that the manipulations above were justified because we had a series of positive real numbers whose sum converged. Just to reinforce this, here's a case in which re-arrangements of the sum are clearly not allowed. Let

$$S = \sum_{i=0}^{\infty}(-1)^i = 1 + (-1) + 1 + (-1) + \cdots.$$

Consider the following:

$$S = [1 + (-1)] + [1 + (-1)] + \cdots = 0 + 0 + \cdots = 0.$$

$$S = 1 + [(-1) + 1] + [(-1) + 1] + \cdots = 1 + 0 + 0 + \cdots = 1.$$

What's the problem here?

If we define $S_n = \sum_{i=0}^{n}(-1)^i$ then $S_n = 1$ for odd n and $S_n = 0$ for even n. Hence S_n does not have a well-defined limit as n tends to infinity. S is not a convergent sum. In general,

5 Team Problems

1. Is a Harvard degree really worth more than an MIT degree?! Let us say that a person with a Harvard degree earns $\$10,000i$ in his/her i^{th} year after graduation, whereas a person with an MIT degree earns $\$1000i^2$ after their i^{th} year. This is before taking inflation into account. Assume inflation is a fixed 8% every year. That is $\$(1+p)$ a year from now is worth \$1 today ($p = 0.08$).

   (a) How much is a Harvard degree holder worth n years after graduation (in terms of today's currency value) ?

   (b) How much is an MIT degree worth in n years?

   (c) Is a Harvard person always worth more? If not, after how many years is an MIT person worth more ?

   [Hint: use either perturbation or differentiation.]

2. Relate the following pairs of functions f, g using asymptotic symbols. E.g. $f(x) = O(g(x))$, $f(x) = \omega(g(x))$. (All logarithms are to base 2.)

   (a) $f(x) = x^{(\log 2m)}$, $g(x) = (2m)^{(\log x)}$

   (b) $f(x) = x^{\log x}$, $g(x) = (\log x)^x$

   (c) For integers n, $f(n) = e^{\mathcal{H}_n}$, $g(n) = \mathcal{H}_{e^n}$.