Tutorial 3

This tutorial will cover three topics. First, we'll see a common fallacy to avoid in inductive proofs called "buildup error". Next, we'll introduce an odd-looking variant on induction called "well-ordering". Finally, we'll prove a theorem about cycles in graphs.

1 Buildup Error

"Buildup error" is a common source of mistakes in inductive proofs. We'll look closely at the anatomy of this error, but remember that this is an error to *avoid*, not a tool to *use*.

Buildup error occurs when the induction hypothesis $P(n)$ is itself an implication of the form $A(n) \Rightarrow B(n)$. In the inductive step of a correct proof, we assume $A(n) \Rightarrow B(n)$ in order to prove $A(n+1) \Rightarrow B(n+1)$. In an erroneous proof, we assume $B(n)$ in order to prove $A(n+1) \Rightarrow B(n+1)$. Here is how the error looks in a complete, false proof:

False Theorem 1 *If every node in a graph is adjacent to a node of the same color, then the graph is monochromatic.*

To see that the theorem is false, consider a cycle on four nodes with two red nodes followed by two blue nodes. Each node is adjacent to some node of the same color, but the graph is not monochromatic.

Proof. The proof is by induction on the number of nodes in the graph. Let $P(n)$ be the predicate, "If every node in an n-node graph is adjacent to a node of the same color, then the graph is monochromatic."

In the base case, $P(1)$, the graph contains only one node and so is certainly monochromatic.

In the inductive step, assume $P(n)$ in order to prove $P(n+1)$. Consider a graph with $n+1$ nodes and pull out one special node. By induction, the remaining n-node graph is monochromatic. Now if the special node is adjacent to a node of the same color, then it must be the same color as all other nodes in the graph. Therefore, the $n+1$-node graph is monochromatic. ■

The flaw is in the statement, "By induction, the remaining n-node graph is monochromatic." This is incorrect. By induction, we can assume only that *if every node in the remaining n-node graph is adjacent to a node of the same color*, then the remaining n-node graph is monochromatic." The correct assumption does not lead to a proof. Good thing, since the theorem is false!

2 Well-Ordering

There is another variation of induction that is sometimes useful: well-ordering. The well-ordering axiom looks nothing like the induction axiom:

Axiom 1 *Every nonempy subset $S \subseteq \mathcal{N}$ has a smallest element.*

The well-ordering axiom is often called the *well-ordering principle.* It may seem obvious and useless!

As for obvious, note that this axiom would be false if the set of non-negative integers, $\mathcal{N}$, were replaced by, say, the set of non-negative rational numbers. The subset S consisting of positive rational numbers has no smallest element.

As for useless, it may be surprising that everything that can be proven by induction can also be proven using the well-ordering principle! In fact the two are interchangeable; we'll see an example in Team Problem 3.

An example of a well-ordering proof is given below. Define a *cycle* in a tournament to be a sequence of players such that each defeats the next and the last defeats the first. The *length* of a cycle is the length of the sequence of players. A *n-cycle* is a cycle of length n. For example, p_1 defeats p_2, p_2 defeats p_3, p_3 defeats p_4, and p_4 defeats p_1 is a 4-cycle.

Theorem 2 *If a tournament contains a cycle, then it contains a 3-cycle.*

Proof. The proof is by contradiction. Assume for the purpose of contradiction that the tournament contains a cycle, but no 3-cycle. By the well-ordering axiom we can let n be the length of the shortest cycle. By assumption $n \geq 3$. Consider three consecutive players in the cycle, p_1, p_2, and p_3. If p_1 defeats p_3, then p_2 can be removed to create a cycle of length $n-1$. This contradicts the assumption that the shortest cycle had length n. If p_3 defeats p_1, then p_1, p_2, and p_3 form a 3-cycle, again a contradiction. ■

3 Hamiltonian Cycles

A *Hamiltonian cycle* of a graph is a cycle that crosses every vertex exactly once. In general, determining whether a graph has a Hamiltonian cycle is difficult.

An example is the problem of finding a *knight's tour.* A chessboard is an 8x8 grid of squares. A knight is a chess piece that moves in "jumps" two squares horizontally and one square vertically or vice versa. If a knight starts in a corner of the board, can it visit every square exactly once and return to its starting position?

We can reduce this to the question of whether a certain graph has a Hamiltonian cycle. In this graph, there is one vertex for each square of the chessboard. There is an edge between two vertices if a knight can jump between the corresponding squares. In fact, this graph does have a Hamiltonian cycle.

The following theorem gives a sufficient condition for a graph to have a Hamiltonian cycle.

Theorem 3 *If every node in an n-node graph has degree at least $n/2$, then the graph has a Hamiltonian cycle.*

Proof. The proof is by contradiction.

Let G be a counterexample to the theorem with a maximal number of edges; that is, if any edge (u, v) is added, then there is a Hamiltonian cycle crossing vertices $u, x_1, x_2, \ldots, x_{n-2}, v$. This implies that G contains a Hamiltonian path from u to v.

Let S be the set of vertices x_i adjacent to v. Since the degree of v is at least $n/2$, $|S| \geq n/2$.

Let T be the set of vertices x_i such that x_{i+1} is adjacent to u. Since the the degree of u is at least $n/2$, u must be adjacent to at least $n/2$ vertices x_i. This implies $|T| \geq n/2 - 1$.

Since S and T are subsets of the $n - 2$ vertices $x_1, \ldots, x_{n-2}$, S and T must share at least one vertex. That is, for some $0 \leq i \leq n - 3$, u is adjacent to x_{i+1} and v is adjacent to x_i. This implies that G contains a Hamiltonian cycle crossing vertices $u, x_1, x_2, \ldots, x_i, v, x_{n-2}, x_{n-3}, \ldots, x_{i+1}, u$.

This contradicts the assumption that G was a counterexample.

Team Problems

1. A *planar graph* is a graph that can be drawn in a plane so that no edges cross. The number of *faces* of a planar graph is the number of regions into which the plane is divided by a drawing of the graph. (The open space surrounding the graph is also considered a face.) Suppose that G is a planar graph. Let V be the number of vertices, E be the number of edges, and F be the number of faces. Prove Euler's formula:

$$E = V + F - 2$$

2. Prove that a graph is bipartite iff it does not contain an odd cycle.

3. Prove Theorem 2 by induction. (Everything that can be proven with the well-ordering principle can be proven by induction.)