Tutorial 1.

1 Predicate logic

Consider the statement, "Every cloud has a silver lining". To say this more formally, we let the variable x range over the set of all clouds, the *universe of discourse*. Then we can write,

$$\forall x : x \quad has \quad a \quad silver \quad lining$$

Here the symbol $\forall$ means "for every" or "for all". Another way to say the same is to let S be the set of clouds. Then this statement can be rewritten:

$$\forall x \in S : x \quad has \quad a \quad silver \quad lining$$

And less formally we can say,

$$\forall cloud \quad x : x \quad has \quad a \quad silver \quad lining$$

Other examples : "Every child is a boy or a girl", can be restated as:

$$\forall \text{ child } x : (x \text{ is a boy}) \vee (x \text{ is a girl})$$

Let x stand for a boolean (0-1 or True/False) variable. The symbol $\vee$ means "or". Then the following statement is always true. (Such a statement is called a "tautology".)

$$\forall x : (x \vee \overline{x})$$

The symbol $\exists$ means "there exists" or "there is at least one". So, for example, "There is always an easy way out" is the same as:

$$\exists \text{ way } x : (x \text{ is easy}) \wedge (x \text{ is a way out})$$

The symbol $\wedge$ means "and". As a second example, let x, y be boolean variables. Then we could say,

$$\exists x, y : (x \vee y) \wedge (x \vee \overline{y})$$

These symbols, $\exists$ and $\forall$, are called *quantifiers*. They can be used together (and also repeatedly). For example, "There is a tallest man" can be written as:

$\exists$ man x $\forall$ man y : x is at least as tall as y

Or "Every tennis tournament has a winner" is:

$\forall tournament \quad x \quad \exists player \quad y : y \quad wins \quad x$

Notice that the order of the quantifiers can change the meaning of the statement. For example the above statement with quantifiers reversed is:

$\exists player \quad y \quad \forall tournament \quad x : y \quad wins \quad x$

which means that the same person wins every tournament whereas all the earlier version meant was that each tournament has a winner, not necessarily the same person. When translating these cryptic statements back to English, it is sometimes helpful to insert a "such that". For example, the preceding statement might be read as, "There exists a player y such that for all tournaments y, y wins x."

What happens to quantifiers when they are negated? We represent "not for every" as $\overline{\forall}$ or $\neg\forall$ and "there does not exist" as $\overline{\exists}$ or $\neg\exists$.

The following statement is a tautology:

$$\overline{\exists} x : (x \wedge \overline{x})$$

That is there is no x for which both x and the negation of x are true. In other words for every x one of x and $\overline{x}$ is false and the other is true. So we could rewrite the above as

$$\forall x : \overline{x \wedge \overline{x}}$$

which is the same as

$$\forall x : \overline{x} \vee x$$

What we did above can be expressed more generally. Let P be any predicate (possibly "quantified"; that is, with some quantifiers in it). Then $\overline{\exists} x : P$ is equivalent to $\forall x : \overline{P}$. and similarly, $\overline{\forall}$ is the same as $\exists x : \overline{P}$. Therefore, every statement with a negated quantifier can be rewritten with the other quantifier and with the predicate negated.

What is the negation of the statement "Every dog has its day"? We can express the negation as

$\overline{\forall} dog \quad x$: x has its day.

Then using the rule above, this can be rewritten as

$\exists dog \quad x : x$ does not have its day.

In other words there is a dog that does not have its day.

A more involved example is to show that:

$$\overline{\forall} x, y \exists z \forall w \exists v : (x \wedge v \wedge \overline{y}) \vee \overline{(x \wedge y \wedge z)}$$

is the same as:

$$\exists x, y \forall z \exists w \forall v : (\overline{x} \vee \overline{v} \vee y) \wedge (x \wedge y \wedge z)$$

The idea here is that the negation causes all of the quantifiers to switch to the other kind. The two predicates are opposites, though this might not be obvious immediately.

Note that:

$$\neg(\exists x \forall y \ldots) = \overline{\exists} x \forall y \ldots$$

2 Proofs about Sets

In this section we carefully prove the following identity. Below A, B, C are finite sets.

Lemma 1 $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

This is known as the *distributive law of set intersection* and will be useful later in the course.

Proof. Let $X = A \cap (B \cup C)$ and $Y = (A \cap B) \cup (A \cap C)$. Then we will show that

1. $X \subseteq Y$ *and*
2. $Y \subseteq X$.

This will show that X and Y must be the same set.

1. $X \subseteq Y$.

   Let x be any element of X. Then $x \in A$ and $x \in (B \cup C)$.

   That is x is in A and also in B or in C. Let us examine these cases separately:

   (a) x is in B. Then x is in $A \cap B$.

   (b) x is in C. Then x is in $A \cap C$.

Therefore x is in $A \cap B$ or in $A \cap C$. That is x is always an element in their union, $x \in (A \cap B) \cup (A \cap C)$.

2. $Y \subseteq X$.

   Let $x \in Y$, i.e., $x \in (A \cap B) \cup (A \cap C)$. There are two cases:

   (a) x is in $A \cap B$. Then $x \in A$ and $x \in B$.

   (b) x is in $A \cap C$. Then $x \in A$ and $x \in C$.

   Therefore x is in A, and x is in $B \cup C$ That is x is in $A \cap (B \cup C)$.

   $x \in X$.

   ∎

3 Proof by Contradiction

Lemma 2 *For any integers m and n, $\sqrt[m]{n}$ (the m^{th} root of n) is either an integer or irrational.*

For example, the lemma tells us that $\sqrt{2}$, $4^{1/3}$, $10^{1/5}$ are all irrational numbers, since clearly none of them are integers.

Proof.

Every real number is either an integer, or a ratio of two integers with the denominator greater than 1, or irrational. Let us assume for the purpose of getting a contradiction that for some integers m, n the number $n^{1/m}$ is of the second kind, i.e., it is the ratio of two integers with the denominator greater than 1.

$$n^{1/m} = \frac{a}{b}$$

for some integers a, b which have no common divisor greater than 1 and $b \neq 1$.

Raising both sides to the power m and rearranging terms we obtain

$$a^m = nb^m$$

Let p be a prime divisor of b (such a prime divisor exists since $b > 1$). Then a^m must also be divisible by the same prime. By the fundamental theorem of arithmetic (unique factorization theorem), this means that p is a divisor of a. But this contradicts our assumption that a, b have no common divisors, this is a contradiction. ∎

Using the same idea as above we can prove the following more general statement.

Lemma 3 *Let the coefficients of the polynomial* $a_0 + a_1x + a_2x^2 + \ldots + x^n = 0$ *be integers. Then any root of the polynomial is either integral or irrational.*

Proof. There are 3 possibilities for a root x of the polynomial. It is either integral, the ratio of two integers with the denominator greater than 1 or irrational. Let us assume for a contradiction that $x = s/t$ for integers s and t that have no common divisors and such that $t > 1$.

Substituting s/t for x and multiplying both sides by t^n gives:

$$a_0t^n + a_1st^{n-1} + a_2s^2t^{n-2} + \ldots + s^n = 0$$

Let p be a prime divisor of t. Then p divides s^n and therefore p divides s. Since we assumed s, t have no common divisors this is a contradiction. ∎

4 Proof by Cases

We have already seen proofs involving cases (e.g., the proof of the distributive law). Now we look at another problem that involves cases. The key to such proofs is to make sure that we cover *all* the cases.

You are given 3 identical looking balls, and told that one of them is heavier than the other two. Can you find the heavy ball using one weighing of a physical balance?

Solution. Place one ball on each pan of the balance and leave the third aside. There are 3 possibilities:

1. The left pan is heavier. Then the ball in the left pan is the answer.
2. The right pan is heavier. Then the ball in the right pan is the answer.
3. The pans are equal. Then the third ball must be the heavy one!

5 Team Problems

Now we'll work out some tougher problems.

1. A hundred soldiers are standing in an array of 10 columns and 10 rows. Let T_i be the tallest soldier in the i^th column and S_j the shortest soldier in the j^th row (i, j run from 1 to 10). Who is taller: the tallest of the S_j's or the shortest of the T_i's? Prove your answer taking care of all the cases.
2. Now you are given 12 identical looking balls, and told that one of them is of different weight from the other 11, but you don't know in advance whether it is heavier or lighter. Can you find the ball that is of different weight using only 3 weighings?