Tutorial 6.

1 Akra-Bazzi

The Akra-Bazzi theorem is easy to apply and quite amazing in its generality. Before it was known however, another fairly general theorem called the *Master theorem* was already quite popular. Most courses in discrete mathematics use that theorem. The Master theorem can be used to solve any recurrence of the form:

$$T(n) = aT(n/b) + n^{\alpha}$$

where $a > 0$ and $b > 1$.

It is a special case of the Akra-Bazzi theorem, so let us see what asymptotic bounds we can get for this type of recurrence using the Akra-Bazzi theorem.

First we find p such that $a(\frac{1}{b})^p = 1$. That is $a = b^p$ or $p = \log_b a$.

Then from the theorem,

$$T(n) = \theta(n^p(1 + \int_1^n \frac{u^{\alpha}}{u^{p+1}} du))$$

$$= \theta(n^p(1 + \int_1^n u^{\alpha-p-1} du))$$

In the case when $\alpha \neq p$,

$$= \theta(n^p(1 + \left[\frac{u^{\alpha-p}}{\alpha - p}\right]_1^n))$$

$$= \theta(n^p(1 + \frac{1}{\alpha - p}(n^{\alpha-p} - 1))) = \theta((1 - \frac{1}{\alpha - p})n^p + \frac{1}{\alpha - p}n^{\alpha})$$

Assuming a, b, α are fixed constants (i.e., don't change with n),

$$T(n) = \theta(n^p + n^{\alpha}) = \theta(n^{\log_b a} + n^{\alpha})$$

If $\alpha = p$, then

$$T(n) = \theta(n^p(1 + \int_1^n \frac{1}{u} du))$$

$$= \theta(n^p(1 + \ln n)) = \theta(n^{\alpha} \log n)$$

So all together we have,

$$T(n) = \begin{cases} \theta(n^{\log_b a}) & \text{if } \log_b a > \alpha \\ \theta(n^{\alpha} \log n) & \text{if } \log_b a = \alpha \\ \theta(n^{\alpha}) & \text{if } \alpha > \log_b a \end{cases}$$

2 Change of variable

Another useful method to make a difficult looking recurrence easier is by a change of variable. This is similar to the change of variable technique in calculus. For example, consider the recurrence

$$T(n) = 2T(\lfloor \sqrt{n} \rfloor) + \log n$$

with base condition $T(1) = 1$. Here $T(n)$ is defined for all n. Let us first note that $T(n)$ is a non-decreasing function of n.

Claim 1 *For all $n > 1$, $T(n) \geq T(n-1)$.*

Proof. By strong indcution on n.
I.H.: $P(n) : T(n) \geq T(n-1)$.
Base: $n = 2$: $T(2) = 2T(\lfloor \sqrt{2} \rfloor) + 1 = 2T(1) + 1 = 3 > 1 = T(1)$.
Ind. Step: Assume $P(k)$ is true for all $k \leq n$. We will prove that $P(n+1)$ is true.

$$\begin{aligned} T(n+1) &= 2T(\lfloor \sqrt{n+1} \rfloor) + \log(n+1) \\ &\geq 2T(\lfloor \sqrt{n} \rfloor) + \log n \end{aligned}$$

The latter is true because either $\lfloor \sqrt{n+1} \rfloor = \lfloor \sqrt{n} \rfloor$ or we can use $P(\lfloor \sqrt{n+1} \rfloor)$. But this is $T(n)$. So $T(n+1) \geq T(n)$. ■

We will now solve the recurrence assuming n is a power of 2. First, make the substitution $n = 2^m$. The recurrence becomes

$$T(2^m) = 2T(\lfloor 2^{m/2} \rfloor) + m$$

Now define $S(m) = T(\lfloor 2^m \rfloor)$ for all integers $m \geq 0$. Note that for all $m > 1$, $S(m+1) \geq S(m)$ since T is non-decreasing, and hence S is also a non-decreasing function.

Therefore,

$$\begin{aligned} S(m) &= T(\lfloor 2^m \rfloor) \\ &\leq T(2^m) \\ &\leq 2T(\lfloor 2^{m/2} \rfloor) + m \\ &\leq 2T(2^{\lceil m/2 \rceil}) + m \\ &\leq 2S(\lceil m/2 \rceil) + m \end{aligned}$$

That is $S(m) \leq 2S(\lceil m/2 \rceil) + m$ with $S(0) = 1$.

We can then view this as $S(m) \leq 2S(m/2 + h(m)) + m$ where $h(m) = \lceil m/2 \rceil - m/2$. Clearly $h(m) < 1$ and hence it is certainly $h(m) = O(\frac{x}{(\log x)^2})$.

Now to solve this we can apply the special case of the Akra-Bazzi theorem from the previous section to get $S(m) = O(m \log m)$.

From which we have that

$$T(n) = O(\log n \log\log n)$$

What about a lower bound? This can be derived in a similar fashion.

Therefore,

$$\begin{aligned} S(m) &= T(\lfloor 2^m \rfloor) \\ &= T(2^m) \\ &= 2T(\lfloor 2^{m/2} \rfloor) + m \\ &\geq 2T(2^{\lfloor m/2 \rfloor}) + m \\ &\geq 2S(\lfloor m/2 \rfloor) + m \end{aligned}$$

That is $S(m) \geq 2S(\lfloor m/2 \rfloor) + m$.

Define a new recurrence $S'(m) = 2S'(\lfloor m/2 \rfloor) + m$ with $S'(0) = 1$.

Claim 2 $S(m) \geq S'(m)$

Proof. By Induction on m.

I.H. : $P(m) : S(m) \geq S'(m)$.

Base : $m = 1$: $S(0) = S'(0) = 1$.

Ind. Step: Assume $P(k)$ is true for all $k \leq m$. We will prove $P(m+1)$.

$$\begin{aligned} S(m) &\geq 2S(\lfloor m/2 \rfloor) + m \\ &\geq 2S'(\lfloor m/2 \rfloor) + m \\ &\geq S'(m) \end{aligned}$$

∎

Now we can apply Akra-Bazzi to solve the recurrence for S' in a similar way as for the upper bound to get $S'(m) = \theta(m \log m)$. From the previous claim this implies that $S(m) = \Omega(m \log m)$.

Hence $T(n) = \Omega(\log n \log\log n)$.

From the upper and lower bounds we conclude that $T(n) = \theta(\log n \log\log n)$.

The hard part is guessing what substitution to make, but this is a skill that comes out of practice, just like in calculus.

3 Team Problems

1. *Life and Death.* In ancient Mars, cannibalism was an essential way of life. The Marons and the Batons were two species that lived for several thousand years in close contact. Every year, first all the Marons sustained themselves by first consuming Batons: two Marons ate one Baton between them. Then they reproduced: Every two one-year-old Marons produced a child and every two of those two years or older produced 3 children. Then the surviving Batons who were two or older consumed one Maron each (one year old Batons didn't need to eat), and then all of them (including surviving one-year olds) reproduced to quadruple in number. This cycle repeated every year. In the beginning there was a single Maron and a single Baton. At the end of the first year there two Marons and two Batons. How many of each will there be at the end of n years?

2. *Stepping up.* Alice the "elegant" climbs up a flight of stairs one or two stairs in one stride. Bob the "elongated beast" on the other hand clambers up 2, 4, 6 or any even number of stairs in one stride. Who can climb a flight of n stairs in more ways? (Assume n is even. A *way* of climbing a flight of n stairs is the sequence of strides taken to get to the top. E.g. to climb up 4 stairs, Alice can go 1,1,1,1 or 1,1,2, or 1,2,1, or 2,1,1 or 2,2 i.e., in 5 different ways in all.)