Massachusetts Institute of Technology
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Lecture 16
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Notes 16

Readings: Section 4-3, 5-4
Next: 4-6, 4-7

Last time, we introduced the *binomial coefficients*

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

These represent the number of ways to choose r elements from a set of n elements, where the order does not matter.
Combinations, as opposed to *permutations*, where the order does matter.
(There are

$$\frac{n!}{(n-r)!}$$

of those.)

Today, we study basic properties of binomial coefficients,
Then an important theorem called the *Binomial Theorem*,
And finally revisit the principle of inclusion and exclusion, using binomial coefficients to prove it.

1 Basic Properties of Binomial Coefficients

Let n and r be naturals, $0 \leq r \leq n$.
Then

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Special cases:

$$\binom{n}{1} = n$$

$$\binom{n}{0} = 1$$

(uses the convention that $0! = 1$),

$$\binom{n}{n} = 1$$

(note symmetry with previous),

$$\binom{n}{n-1} = n$$

(symmetry).

There are infinitely many interesting identities involving binomial coefficients.
You can't learn them all, but you can learn the basic ones and look others up/derive them as needed.
Can prove them by algebra, or by "thought experiments" that establish a combinatorial meaning for the LHS and RHS and argue that they are the same.

Theorem 1.1 Suppose $0 \leq r \leq n$.
Then

$$\binom{n}{r} = \binom{n}{n-r}.$$

Proof. (Algebraic)

$$\frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!(n-(n-r))!}.$$

■

Proof. (Combinatorial thought experiment)
LHS is number of ways to choose r from n.
RHS is number of ways to choose $n-r$.
Choosing a set of r is equivalent to choosing a set of $n-r$
(the elements *not* in the first chosen set)
– so there are just as many ways to choose either. ■

The next theorem gives the ratio between successive binomial coefficients (same n, successive r):

Theorem 1.2 Suppose $1 \leq r \leq n$.
Then

$$\binom{n}{r} / \binom{n}{r-1} = (n-r+1)/r.$$

Proof. (Algebraic)
LHS =

$$\frac{n!}{r!(n-r)!} \times \frac{(r-1)!(n-(r-1))!}{n!} = (n-r+1)/r.$$

■

Note that this ratio is > 1, which means the successive coefficients are increasing, as long as $r < \frac{n+1}{2}$.
Also, the ratio is < 1, which means the coefficient are decreasing, when $r > \frac{n+1}{2}$.
This says that the coefficients increase, then decrease, i.e., they are *unimodal.*
Form a bell-shaped curve:

Figure 1: Bell Curve

In fact, the binomial coefficients form a real *bell curve.*
Like on tests:
Suppose that an SAT test has n questions.
Then the bell curve describes, for every r, the number of ways the student can get exactly r of the questions right, that is,

$$\binom{n}{r}.$$

If we suppose that each student taking the test is equally likely to get each of the n questions right or wrong, $\frac{1}{2}$ for each, then these coefficients turn out to be proportional to the number of students that get those numbers of questions right.
(See this when we do probability.)

Another identity:

Theorem 1.3 Suppose $1 \leq r \leq n$.
Then

$$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}.$$

Proof. (Algebraic)

$$\begin{aligned}\binom{n}{r} &= \frac{n!}{r!(n-r)!} \\ &= \frac{n}{r}\frac{(n-1)!}{(r-1)!(n-r)!} \\ &= \frac{n}{r}\frac{(n-1)!}{(r-1)!((n-1)-(r-1))!} \\ &= \frac{n}{r}\binom{n-1}{r-1}\end{aligned}$$

■

Proof. (Combinatorial)
Consider choosing a committee of size r and a leader, from n people.
One way is to pick the leader first (n ways), then pick the other $r-1$ committee members from the remainder ($\binom{n-1}{r-1}$ ways).
Another way is to pick the r committee members first ($\binom{n}{r}$ ways) and then pick a leader from among them (r ways).
This implies that:

$$n\binom{n-1}{r-1} = \binom{n}{r}r.$$

Now divide both sides by r. ■

Now an important theorem due to Pascal:

Theorem 1.4 (Pascal) Suppose $1 \leq r \leq n-1$.
Then

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$

Proof. (Algebraic)

$$\begin{aligned}\binom{n-1}{r} + \binom{n-1}{r-1} &= \frac{(n-1)!}{r!(n-1-r)!} + \frac{(n-1)!}{(r-1)!(n-r)!} \\ &= (n-r)\frac{(n-1)!}{r!(n-r)!} + r\frac{(n-1)!}{r!(n-r)!} \\ &= n\frac{(n-1)!}{r!(n-r)!} \\ &= \frac{n!}{r!(n-r)!} \\ &= \binom{n}{r}\end{aligned}$$

■

Proof. (Combinatorial) Let $S = \{1, \ldots, n\}$.
Let A be the set of r-element subsets of S.
Let B be the set of r-element subsets of S that contain 1.
Let C be the set of r-element subsets of S that don't contain 1.

Then $A = B \cup C$, and B and C are disjoint.
So $|A| = |B| + |C|$, by sum rule.
But now can get expressions for $|A|$, $|B|$ and $|C|$ as numbers of combinations:

- $|A| = \binom{n}{r}$.
- $|B| = \binom{n-1}{r-1}$.
  This is because $r-1$ other elements must be chosen from the remaining $n-1$.
- $|C| = \binom{n-1}{r}$.
  This is because r elements must be chosen from the remaining n.

So

$$\binom{n}{r} = |A| = |B| + |C| = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

■

Pascal's theorem has a nice pictorial representation:
Row represents n, starting with 0 in top row.
Successive elements represent r, starting with 0 at left.

Figure 2: Pascal's triangle.

This triangle has lots of nice properties.
Experiment with it.
E.g., what happens if we add the coefficients in one row?

Theorem 1.5 Suppose n is any natural.
Then

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n.$$

Proof. (Algebraic)
Induction on n.
Let $P(n)$ be:

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n$$

BASE: $n = 0$.

$$\sum_{r=0}^{0} \binom{0}{r} = \binom{0}{0} = 1 = 2^0$$

INDUCTIVE STEP: Assume $P(n)$, show $P(n+1)$, where $n \geq 0$.

$$\begin{aligned}
\sum_{r=0}^{n+1} \binom{n+1}{r} &= \tbinom{n+1}{0} + \textstyle\sum_{r=1}^{n} \tbinom{n+1}{r} + \tbinom{n+1}{n+1} \\
&= 2 + \sum_{r=1}^{n} \binom{n+1}{r} \\
&= 2 + \sum_{r=1}^{n} \left(\binom{n}{r} + \binom{n}{r-1} \right) \qquad \text{by Pascal's identity} \\
&= 2 + 2\sum_{r=1}^{n-1} \binom{n}{r} + \binom{n}{0} + \binom{n}{n} \\
&= 2\sum_{r=0}^{n} \binom{n}{r} \\
&= 2 \cdot 2^n \qquad \text{by inductive hypothesis,} \\
&= 2^{n+1}.
\end{aligned}$$

■

Proof. (Combinatorial)
There are 2^n different subsets of a set of n elements.
Decompose this collection based on the sizes of the subsets.
There are $\binom{n}{r}$ subsets of size r, for $r = 0, 1, \ldots, n$.
Hence the theorem follows. ■

Theorem 1.6 (Vandermonde) Suppose $0 \leq r \leq m, n$.
Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}$$

Proof. (Combinatorial)
Suppose there are m red balls and n blue balls, all distinct.
There are $\binom{m+n}{r}$ ways to choose r balls from the two sets combined.
That's the LHS.
Now decompose this collection of choices based on how many of the chosen balls are blue.
For any $k, 0 \leq k \leq r$, there are $\binom{m}{r-k}$ ways to choose $r-k$ red balls, and $\binom{n}{k}$ ways to choose k blue balls.
By the product rule, that makes $\binom{m}{r-k}\binom{n}{k}$ ways to choose r balls such that k of them are blue.
Adding up these numbers for all k gives the RHS. ■

It is also possible to prove this algebraically, using Pascal's identity.
To do this, it's useful to generalize the notion of binomial coefficient a little bit:
If $r > n$, define $\binom{n}{r}$ to be 0.
Makes some kind of sense informally, because there aren't any ways to choose more elements than there are in a set!

We can then observe that some of the binomial identities are still correct with this extension.
E.g., Pascal's theorem is still true if $r > n$ (reduces trivially).
So we will prove Vandermonde's theorem without the restriction $r \leq m, n$.
This means that some of the terms in the statement of the theorem could be trivial terms, e.g., $\binom{n}{k}$ could occur, where $k > n$ (value taken as 0).

Proof. (Algebraic)
Induction using Pascal's identity:
Let $P(n)$ be

$$\forall m, r, \binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

BASE: $n = 0$. Then

$$\binom{m+0}{r} = \binom{m}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{0}{k}.$$

This latter equation is true because the only value of k for which the last factor isn't 0 is $k = 0$.
Hence $P(0)$ is true.

INDUCTIVE STEP: Assume $P(n)$ is true, show $P(n+1)$. We have:

$$\begin{aligned}
\binom{m+(n+1)}{r} &= \binom{m+n}{r} + \binom{m+n}{r-1} && \text{by Pascal's identity (don't have to worry about bounds)} \\
&= \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k} + \sum_{k=0}^{r-1} \binom{m}{r-1-k}\binom{n}{k} && \text{by inductive hypothesis (don't have to wor}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k} + \sum_{k=1}^{r} \binom{m}{r-k}\binom{n}{k-1} \quad \text{by change of vars} \\
&= \binom{m}{r}\binom{n}{0} + \sum_{k=1}^{r} \binom{m}{r-k}\left(\binom{n}{k} + \binom{n}{k-1}\right) \\
&= \binom{m}{r}\binom{n}{0} + \sum_{k=1}^{r} \binom{m}{r-k}\binom{n+1}{k} \quad \text{by Pascal's identity} \\
&= \binom{m}{r}\binom{n+1}{0} + \sum_{k=1}^{r} \binom{m}{r-k}\binom{n+1}{k} \\
&= \sum_{k=0}^{r} \binom{m}{r-k}\binom{n+1}{k}
\end{aligned}
$$

Hence, $P(n+1)$ is true. So $P(n)$ is true for all n. ■

2 The Binomial Theorem

An important theorem giving the coefficients in the expansion of powers of binomial expressions.
Turns out they are binomial coefficients.

2.1 The Theorem

Theorem 2.1 Suppose $n \in \mathbb{N}$. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Example 1 $(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$

Proof. (Algebraic)
By induction (HW) ■

Proof. (Combinatorial)

$$(x+y)(x+y)(x+y)(x+y)\ldots$$

n factors.
Each $x^k y^{n-k}$ term selects k binomials from the n to have x, rest y.
$\binom{n}{k}$ ways of choosing the binomials to have x. ■

2.2 Identities via Binomial Theorem

The binomial theorem can be used to give slick proofs of some binomial identities, e.g.:

Theorem 2.2

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof. Plug $x = 1$, $y = 1$ into binomial theorem. ∎

Theorem 2.3

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Proof. Plug $x = -1$, $y = 1$ into binomial theorem. ∎

Combinatorial interpretation: number of ways of selecting an even number of elements from a set of n equals the number of ways of selecting an odd number.
This may be somewhat surprising – works whether n is even or odd.

Example 2 $n = 5$:
Choosing odd is $\binom{5}{1} + \binom{5}{3} + \binom{5}{5} = 5 + 10 + 1 = 16$.
Choosing even is $\binom{5}{0} + \binom{5}{2} + \binom{5}{4} = 1 + 10 + 5 = 16$.

Example 3 $n = 6$:
Choosing odd is $6 + 20 + 6 = 32$.
Even is $1 + 15 + 15 + 1 = 32$.

Theorem 2.4

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} = 2^{n-1}.$$

Proof. (Algebraic)
Add the equations in the two previous theorems.
Get

$$\sum_{k=0}^{n} (\binom{n}{k} + (-1)^k \binom{n}{k}) = 2^n.$$

Odd terms cancel out, even terms double, so get:

$$2\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} = 2^n.$$

Then divide by 2. ■

Proof. (Combinatorial)
LHS represents number of ways to pick an even number of elements from a set of n elements. The number of ways to pick an even number is equal to the number of ways to pick an odd number of elements.
Since the total number of ways is 2^n, the number of ways to pick even is half this, or 2^{n-1}. ■

A new proof of Pascal's identity:

Proof. $(x+y)^n = x(x+y)^{n-1} + y(x+y)^{n-1}$.
Consider coefficient of $x^r y^{n-r}$.
LHS: $\binom{n}{r}$.
RHS: Obtained by adding coefficient of $x^{r-1}y^{n-r}$ in the first term, plus the coefficient of $x^r y^{n-1-r}$ in the second term.
These are $\binom{n-1}{r-1}$ and $\binom{n-1}{r}$, respectively. ■

Vandermonde:

Proof. $(1+t)^{m+n} = (1+t)^m(1+t)^n$.
Consider coefficient of t^r.
LHS: $\binom{m+n}{r}$.
RHS: Obtained by adding up, for all k, the product of the coefficient of t^k in the second factor and the coefficient of t^{r-k} in the first factor.
This is

$$\Sigma_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}.$$

■

2.3 General Binomial Theorem

Can generalize binomial coefficients once again, allowing the first argument to be any real number:

Definition 2.5 General binomial coefficient $\binom{\alpha}{k}$ for $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$:

$$\binom{\alpha}{k} = \tfrac{1}{k!}\textstyle\prod_{j=0}^{k-1}(\alpha - j)$$

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

$$\binom{\alpha}{0} = 1$$

This doesn't have the same kinds of combinatorial meaning as before, but works out nice mathematically.
In particular, a version of the binomial theorem holds:

Theorem 2.6 Let $\alpha \in \mathbb{R}$ and $\left|\frac{x}{y}\right| < 1$. Then

$$(x+y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha-k}$$

Proof. Calculus. ∎

Example 4 Approximate $\sqrt{2}$ using binomial theorem.

$$\sqrt{2} = (x+y)^{\frac{1}{2}}, \text{ where } x = 1, y = 1$$

However, can't use b.t. directly, because

$$\left|\frac{x}{y}\right| \not< 1$$

.
However,

$$\sqrt{2} = 2\frac{\sqrt{2}}{2} = 2\sqrt{\frac{1}{2}} = 2(1-\frac{1}{2})^{\frac{1}{2}} = 2(x+y)^{\frac{1}{2}}, \text{ where } x = 1, y = -\frac{1}{2}$$

$$(1-\frac{1}{2})^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \left(-\frac{1}{2}\right)^k$$

$$\binom{\frac{1}{2}}{0} = 1, \binom{\frac{1}{2}}{1} = \frac{1}{2}, \binom{\frac{1}{2}}{2} = \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!} = -\frac{1}{8}, \binom{\frac{1}{2}}{3} = \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!} = \frac{1}{16}, etc.$$

$$\begin{aligned}\sqrt{2} &= 2\left(1 + \tfrac{1}{2}\left(-\tfrac{1}{2}\right) + \left(-\tfrac{1}{8}\right)\left(\tfrac{1}{4}\right) + \left(\tfrac{1}{16}\right)\left(-\tfrac{1}{8}\right) + \cdots\right) \\ &= 2\left(1 - \tfrac{1}{4} - \tfrac{1}{32} - \tfrac{1}{128} - \cdots\right) \\ &\approx 2(.7109375) = 1.421875\end{aligned}$$

3 Principle of Inclusion and Exclusion

Finally, we return to the principle of inclusion-exclusion from last lecture.
Use binomial coefficients to help prove this.
Recall (for general m):

Theorem 3.1 $|\cup_i A_i| = \Sigma_i |A_i| - \Sigma_{i,j \text{ distinct}} |A_i \cup A_j| + \Sigma_{i,j,k \text{ distinct}} |A_i \cup A_j \cup A_k| - \ldots + (-1)^{m+1} |\cap A_i|$

Can be proved by induction on the number of sets – LTTR.
Now we will give a combinatorial proof using binomial coefficients.

Proof. (Combinatorial)
Each term in the summation counts certain elements in $\cup_i A_i$.
We prove that every element of this union is counted exactly once.
So, consider any particular element a, and suppose it is in exactly r of the sets A_i.
We see how many times a is counted by each term.

In the first term, a is counted $r = \binom{r}{1}$ times, once for each of the sets that contains it.
In the second term, a is counted $\binom{r}{2}$ times, once for each of the pairs of sets that contain it.
In the third term, a is counted $\binom{r}{3}$ times, ...,
and in the rth term, a is counted $\binom{r}{r} = 1$ times.
The terms alternate signs, so the total number of times a is counted is:

$$\binom{r}{1} - \binom{r}{2} + \binom{r}{3} - \ldots - (-1)^r \binom{r}{r}.$$

But Theorem 2.3 implies that

$$\sum_{k=0}^{r} (-1)^k \binom{n}{k} = 0.$$

This implies that the sum above is equal to $\binom{r}{0} = 1$.
That is, a is counted exactly once, as needed. ∎