Problem Set 11 Solutions

Problem 1. Total number of hands is $\binom{52}{13}$. To count the number of hands with 9 cards of the same suit, we need to (1) choose the suit, (2) choose the 9 cards of the same suit, and (3) choose the remaining 4 cards. This can be done in $4\binom{13}{9}\binom{39}{4}$ ways. Note, there is no possible double counting, since a hand can have at most one 9-carded suit. Thus, the probability is

$$\frac{4\binom{13}{9}\binom{39}{4}}{\binom{52}{13}} = 0.0003704$$

The number of hands with exactly 5 spades is $\binom{13}{5}\binom{39}{8}$, which is the same as the number of hands with exactly 5 hearts. However, we double counted. The number of hands with both 5 spades and 5 hearts is $\binom{13}{5}\binom{13}{5}\binom{26}{3}$. Thus, the probability is

$$\frac{2\binom{13}{5}\binom{39}{8} - \binom{13}{5}\binom{13}{5}\binom{26}{3}}{\binom{52}{13}} = 0.2426$$

Problem 2.

(a) For a given ticket, it will win if any of $\binom{20}{10}$ sets of numbers are choosen. Since there are $\binom{80}{10}$ equally likely sets of numbers, the probability of winning is

$$\frac{\binom{20}{10}}{\binom{80}{10}} = 0.0000001122$$

We will denote this probability by $\mathcal{P}$.

(b) For the lower bound, if all of the students select the same set of numbers, then the probability of winning is the same as above. Thus, the lower bound is $\mathcal{P}$.

For the upper bound, the probability of any student winning is at most the sum of probabilities of each student winning, which is $142\mathcal{P}$.

(c) Here, we are only concerned about the number of distinct tickets. For an upper bound, the probability of a win is at most the sum of the probabilities of each distinct ticket of winning – $k\mathcal{P}$.

The lower bound is more difficult to analyze. We need to analyze the number of distinct sets of ten numbers. A simple lower bound is probability of a single ticket winning – $\mathcal{P}$.

(d) All the students could have choosen the same ticket. In that case, if one student wins, they all win. Thus, a lower bound on the probability of exactly one winner is zero. On the other hand, the tickets could be arranged so that no two different tickets have the same 10 numbers choosen. In that case, if exaclty one person can win at any given time, the probability of a win is as in (b), $142\mathcal{P}$.

(e) Let s be the number of distinct tickets chosen by only one person (i.e. $p_i = 1$). For there to be exactly one winner, one of these s tickets must be chosen and no other ticket wins. This probability has an upper bound of $s\mathcal{P}$. For the lower bound, it is possible that every set of 10 numbers appears on two different tickets. Thus, the lower bound is zero.

Problem 3.

(a) Out of the n tosses, k of them are heads and $n-k$ of them are tails. The probability of any particular sequence is $p^k(1-p)^{n-k}$. For a sequence of n coins, there are $\binom{n}{k}$ ways to pick a sequence with k heads. Thus, the probability of exactly k heads is

$$\binom{n}{k} p^k (1-p)^{n-k}$$

(b) Consider the ratio of the probability of k heads over the probability of $k-1$ heads.

$$\begin{aligned}
\frac{\binom{n}{k} p^k (1-p)^{n-k}}{\binom{n}{k-1} p^{k-1}(1-p)^{n-k+1}} &= \frac{\frac{n!}{k!(n-k)!}p}{\frac{n!}{(k-1)!(n-k+1)!(1-p)}} \\
&= \frac{(n-k+1)p}{k(1-p)}
\end{aligned}$$

This quantity is greater than 1 if and only if $(n-k+1)p > k(1-p) \iff np+p > k$. So, for $k < np+p$ the probability increases from $k-1$ to k, and for $k > np+p$, the probability decreases. The value of k that maximizes the probability is thus $\lfloor np+p \rfloor$.

(c) The probability of no heads is $(1-p)^n = (1-\frac{1}{n})^n$. For large n, this expression approaches $1/e$.

The probability of exactly one head is $\binom{n}{1}p(1-p)^{n-1} = (1-\frac{1}{n})^{n-1}$. Forming the ratio, we find the probability of no heads over the probability of exactly one head is $\frac{1}{1-1/n}$, which is one over the probability of a tail.

Problem 4.

$$Pr(A \cap \overline{B}) \quad = \quad Pr(A) - Pr(A \cap B)$$

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\begin{aligned}
&= Pr(A) - Pr(A)Pr(B) && \text{since } A \text{ and } B \text{ are independent} \\
&= Pr(A) \cdot (1 - Pr(B)) \\
&= Pr(A) \cdot Pr(\overline{B})
\end{aligned}
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Problem 5.

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\begin{aligned}
Pr(R^2 = r \cap e^T = t) &= Pr[(R = \sqrt{r} \cup R = -\sqrt{r}) \cap (e^T = t)] \\
&= Pr[(R = \sqrt{r} \cup R = -\sqrt{r}) \cap (T = \ln t)] \\
&= Pr[((R = \sqrt{r}) \cap (T = \ln t)) \cup ((R = -\sqrt{r}) \cap (T = \ln t))] \\
&= Pr[(R = \sqrt{r}) \cap (T = \ln t)] + Pr[(R = -\sqrt{r}) \cap (T = \ln t)] \\
&= Pr(R = \sqrt{r})Pr(T = \ln t) + Pr(R = -\sqrt{r})Pr(T = \ln t) \quad \text{since } R\&T \text{ ind.} \\
&= Pr(R^2 = r)Pr(e^T = t)
\end{aligned}
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Problem 6.

(a) $Pr[\geq 0.01] = 1 - Pr[< 0.01] \leq 1 - (\frac{1-0.01}{1-0.02})2^{-2000}\binom{2000}{0.01\times 2000}$

(b) $Pr[\geq 600] = 1 - Pr[< 600](n = 1000, \alpha = 3/5) \leq 1 - 2^{-1000}\binom{1000}{600}\frac{1-3/5}{1-6/5}$

(c) $Pr[\geq 200] = 1 - Pr[< 200](n = 1000, \alpha = 1/5) \leq 1 - 2^{-1000}\binom{1000}{200}\frac{1-1/5}{1-2/5}$

Problem 7.

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\begin{aligned}
Pr(M_1 = x) &= Pr(max(D_1, D_2) = x) \\
&= Pr(D_1 = x \text{ and } D_2 < x) + Pr(D_1 < x \text{ and } D_2 = x) + Pr(D_1 = D_2 = x) \\
&= \frac{1}{6} \cdot \frac{x-1}{6} + \frac{x-1}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} \\
&= \frac{2x-1}{36}
\end{aligned}
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M_1 and M_2 are not independent since we have:

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\begin{aligned}
Pr(M_1 = x | M_2 > x) &= 0 \qquad \text{for } x \in \{1, 2, \ldots 6\} \\
&\neq Pr(M_1 = x)
\end{aligned}
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