Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
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Handout 36
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Practice Problems

(Final Examination, Fall 1995)

Problem 1. Prove **using induction** that the sum of the cubes of three consecutive positive integers is divisible by 9.

Problem 2. For each of the following statements, (1) circle **T** or **F** to indicate whether the statement is true or false, respectively, and (2) justify your answer succinctly. The justification counts for more points than does the correct **T/F** answer. A formal proof is not necessary, but you should be able to write a one- or two-sentence explanation or picture that conveys the essential idea.

- $\forall X \subseteq \mathbb{R}$, if X is uncountable, then $\mathbb{R} - X$ is countable.
- $f(x) = |x|$ is an injection from $\mathbb{R}$ to $\mathbb{R}$.
- $f(x) = x^{1995}$ is a surjection from $\mathbb{R}$ to $\mathbb{R}$.
- For all finite sets A and B, if A is a proper subset of B, then there is no bijection between A and B.
- Let a_n be the probability that you pick the maximum on picking a number uniformly at random from an array of n distinct numbers. Assume $a_0 = 0$. Then the OGF for a_n is $x/(1-x)^2$.

Problem 3. Short answer questions. Justify your answer succinctly.

(a) What is the coefficient of x^3 in the binomial expansion of $(1-x)^{-\frac{1}{2}}$?.

(b) Evaluate the sum

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x^{i-j} y^{i+j}$$

(c) Find the number of three letter words over the alphabet a,b,...,z that do not contain "ab" or "ba".

Problem 4. Let A and B be two equivalence relations on the set $S = \{1, 2, \ldots, n\}$. Prove that $A \bigcap B$ is an equivalence relation too. (Recall that a relation on a set S is a subset of $S \times S$.)

Problem 5. For each of the following pairs of functions $f(n)$ and $g(n)$, fill in the blank with o, O, ω, Ω, and Θ depending on whether $f(n) = o(g(n))$, $f(n) = O(g(n))$, $f(n) = \omega(g(n))$, $f(n) = \Omega(g(n))$, or $f(n) = \Theta(g(n))$. If there is more than one relation between $f(n)$ and $g(n)$, write the strongest one. As usual, $\lg n$ denotes logarithm to the base 2. Each part is worth 2 points.

$$\begin{aligned}
10^{\lg n} &= \underline{\hspace{2cm}}(n^{\lg 10}) \\
(n!)^2 &= \underline{\hspace{2cm}}(n^n) \\
2^{(\lg^3 n)} &= \underline{\hspace{2cm}}(n^{4 \lg \lg n}) \\
\binom{n}{k} &= \underline{\hspace{2cm}}(n^k) \qquad \text{for any constant } k \\
\sum_{i=1}^{n} \frac{1}{i^5} &= \underline{\hspace{2cm}}(1)
\end{aligned}$$

Problem 6. **(a)** Let

$$T(n) = 2T(\lfloor n/4 \rfloor) + T(\lfloor n/3 \rfloor) + 2n$$

for all positive integers n and let $T(0) = 0$. Prove **by induction** that $T(n) = O(n)$.

(b) Solve

$$T(n) = 5T(n-1) - 4T(n-2)$$

for all $n \geq 2$. $T(0) = 3$ and $T(1) = 6$.

Problem 7. **(a)** How many ways can you arrange 39 identical black balls and 13 identical white balls in a row, so that no two white balls are adjacent?

(b) A standard deck of 52 cards is shuffled completely (i.e. any of the permutations are equally likely.) What is the **probability** that no two spades are adjacent in the shuffled deck?

Problem 8. A pitcher's record is 21 wins and 4 losses. **Prove** that he must have enjoyed a winning streak of 5 games.

Problem 9. 10% of the theorems in the lecture notes contain typos. For any theorem, it is the case that if you ask a TA whether it has a typo the probability that her answer is correct is 0.8.

(a) Given a theorem selected uniformly at random, if a TA tells you that it has a typo, what is the probability that it does?

(b) Suppose TAs answer independently from each other. If 2 TAs tell you a theorem has a typo, what is the probability that it does?

Problem 10. An integer X is drawn from the set $1, 2, \ldots, n$ uniformly at random.

(a) Prove that the probability that X is divisible by k is $(1/n)(\lfloor n/k \rfloor)$.

(b) Prove that the expected number of divisors of X is $\Theta(\log_e n)$

Problem 11. **(a)** Let G be a graph with E edges. If each vertex is colored red or blue independently with equal probability then what is the expected number of edges with differently colored endpoints.

(b) Let $N(i, j), 1 \le i, j \le n$ represent the number of times person i calls person j, per month. Calls within an area code cost L, and calls between area codes cost D. If the phone company randomly assigns each person to one of k area codes, then what is its expected monthly revenue?

Problem 12. 100 students take an exam. The total number of points for the exam is 207.

(a) If the mean on the exam is 60 then what is a nontrivial upper bound on the number of students who scored **more than** 120?

(b) If the mean on the exam is 60 and the variance is 72 then what is the best upper bound you can give on the number of students who scored **more than** 120?

Problem 13. An exam has 100 true/false questions and 100 multiple choice questions. Each true/false question is worth 1 point while each multiple choice question is worth 2 points. The maximum score is 300. There is a student who answers at random. She is correct with probability 1/2 on each true/false question and with probability 1/4 on each multiple choice question.

(a) What is the expected score of the student?

(b) What is the standard deviation of the student's score?

(c) What is the best upper bound you can give on the probability that the student scores more than 272 points?