Problem Set 10 Solutions

Problem 1.

The probability of winning the game remains the same but now there is more than one optimal strategy. If Carole reveals the "Nerd Pride Button" it means that you must have chosen the box covering the "Nerd Pocket Protector". In this case, you win with probability 1 if you switch. If Carole shows you the "Nerd Pocket Protector" then there is 1/2 probability that you have picked the prize box, so you win with probability 1/2 whether you switch or not. Let NPB be the event that Carole shows the "Nerd Pride Button", and NPP be the event that Carole shows the "Nerd Pocket Protector". Then the total probability of winning is:

$$\begin{aligned} Pr(win) &= Pr(win \mid \text{NPB})Pr(\text{NPB}) + Pr(win \mid \text{NPP})Pr(\text{NPP}) \\ &= 1 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3} \\ &= \frac{2}{3} \end{aligned}$$

Problem 2.

(a) In this case, you should always pick Box 1. Monty (Prof. Leighton) will be forced to reveal the location of the other "Nerd Pride" pin. Upon switching, you will be guaranteed to win the ice cream (i.e. the probability of winning is 1). If you pick a box besides 1, Monty can always reveal Box 1 (giving you no additional information), and you will have a 1/2 chance of winning.

(b) If you play the "don't-switch strategy", you will win whenever you pick the correct box initially. Thus, you will win with probability 1/3. If you play the "switch strategy", you will lose every time you initially pick the wrong box, since Monty can decide to end the game. Everytime you pick the right box, Monty will reveal a bad box, and then you will switch to a bad box. Thus, you lose again. Therefore, the probability of winning is 0 with the "switch strategy".

Problem 3. Let A_i be the event that componet i fails. Let A be the probability that one or more components fail. Then we have:

$$\begin{aligned} Pr(A) &= \sum_{i=1}^{1000} Pr(A_i) \\ &= 1000 \cdot Pr(A_i) \\ &\leq 1000 \cdot \frac{1}{10,000} \qquad \text{since } Pr(A_i) \leq 1/10,000 \\ &= 1/10 \end{aligned}$$

Problem 4. Let B be the event the criminal has a beard. Let W be the event the witness says the criminal has a beard. We want to compute $Pr(B|W)$. By Bayes's theorem,

$$Pr(B|W) = \frac{Pr(W|B)Pr(B)}{Pr(W|B)Pr(B) + Pr(W|\overline{B})Pr(\overline{B})} = \frac{(0.8)(.15)}{(0.8)(0.15) + (0.12)(0.85)} = 0.54$$

Problem 5. **(a)**

$$\begin{aligned} 1 - Pr(A|C) &= 1 - \frac{Pr(A \cap C)}{Pr(C)} \\ &= \frac{Pr(C) - Pr(A \cap C)}{Pr(C)} \\ &= \frac{Pr(\overline{A} \cap C)}{Pr(C)} \\ &= Pr(\overline{A}|C) \end{aligned}$$

(b) Yes, it is. Consider the uniform sample space $S = \{1, 2, \ldots, 2 \cdot 10^6\}$.
Let $A = \{1\}, B = \{1, 2, \ldots, 10^6\}, C = \{1, 10^6 + 1, \ldots, 2 \cdot 10^6 - 1\}$.
Then $Pr(A \mid B) = Pr(A \mid C) = 10^{-6}$, while $Pr(A \mid B \cap C) = 1$.