Problem Set 12

Due: December 6, 1996 in recitation or by 5 pm in NE43-366.

Reading: Lecture Notes on *Probability*

Problem 1. [12 points]

At the carnival, there is a betting game played with three fair and (mutually) independent dice. You may bet \$1 on any of the six values $\{1, 2, 3, 4, 5, 6\}$. If any of the dice matches your value, you get your original bet back plus you win \$1 if one of the dice matches your value, \$2 if two of the dice match your value, and \$3 if three of the dice match your value.

Bobo thinks the game is fair since if he simultaneously bets \$1 on each of the 6 choices, and the dice come up a, b, c, he wins 3 bets plus \$3, for a total of \$6, which is what he wagered. Bobo figures that he does the same if some value comes up more than once.

(a) What is the probability that

1. All three dice have different values?
2. Two dice have the same value and the third die has a different value?
3. All three dice have the same value?

(b) What is the expected return (net winnings) if you wager \$6, placing \$1 on each of the six values? Justify your answer.

(c) What is the expected return if you wager \$1 on only one of the six values? Justify your reasoning.

(d) Set up this game so that it is fair by fixing the three prizes i_1, i_2, i_3 so that if you match d of the dice, you win $\$i_d$ (plus you get your your original \$1 bet back).

Problem 2. [8 points]

Consider a system consisting of 2 components, each of which can fail at each step with probability p. Failures at different steps and/or components are mutually independent (in particular, this means that failure need not be permanent). The steps are numbered $1, 2, \ldots$.

(a) Let X be the random variable whose value is the step number of the first step at which there is a failure (in either component). Find $\mathbf{E}[X]$.

(b) Let Y be the random variable whose value is the step number of the first step at which both components fail simultaneously. Find $\mathbf{E}[Y]$.

Problem 3. **[10 points]**

The same n college students apply to two graduate schools, A and B. In order to same money and time, each grad school ranks the students randomly, with all permutations chosen with equal likelihood. The rankings of the two graduate schools are independent of each other.

(a) What is the expected number of students that have a higher rank on the list for grad school B than they do for grad school A?

(b) What is the expected number of students that have a rank on the list for grad school B that is at least k higher than their rank on the list for grad school A?

Problem 4. **[20 points]**

Suppose n balls are thrown randomly and mutually independently into n boxes, so each ball lands in each box with uniform probability.

(a) What is the expected number of empty boxes? *Hint:* Use linearity of expectation.

(b) Show that $\mathbf{Pr}[\geq k \text{ balls fall in the first box}] \leq \binom{n}{k}\left(\frac{1}{n}\right)^k$.
Argue that this probability is at most $1/k!$, and therefore
$\mathbf{Pr}[\geq k \text{ balls fall into } \textit{some} \text{ box}] \leq n/k!$.

(c) Let random variable R be the maximum number of balls that land in any box. Use Stirling's formula and the result from the last part to give an upper bound for $\mathbf{Pr}[R \geq k]$.

(d) How does your answer to the last part behave asymptotically with n when $k = n^\alpha$ for some constant α? Is $\mathbf{Pr}[\geq k \text{ balls land in some box}]$ large or small?
What about when $k = \lg \lg n$?

(e) We'd like to know when this upper bound goes from being close to 1 to being close to 0 as we increase k. Show that this happens around $k \approx \dfrac{\lg n}{\lg \lg n}$.
(You only need to find when n/k^k goes from being close to 0 to close to 1.)

Problem 5. **[8 points]**

Consider the following 3-stage experiment:

In the first stage, I throw a 6-sided die. The number of spots that I get determines how many instances of the second stage to perform.

In the second stage, I throw a 6-sided die. This stage is repeated once for each spot I got in the first stage. The total number of spots that I get in this stage determines how many instances of the third stage to perform.

In the third stage, I throw a 6-sided die. This stage is repeated once for each spot I got in all instances of the second stage.

For example, if the outcome of the first stage is 3, then stage 2 gets played 3 times. Suppose the outcomes of the 3 instances of stage 2 are $1, 3, 4$. Then stage 3 is played $1 + 3 + 4 = 8$ times, and the score is the sum of 8 instances of stage 3.

Assume that all throws are mutually independent.

My score is the total number of spots on all the instances of the third stage.

What is my expected score?

(Analyze this as carefully as you can, using Wald's theorem. Try to be careful about uses of independence.)

Problem 6. **[20 points]**

Let X and Y be random variables whose values are independent observations drawn uniformly at random from the set $\{1, \ldots, n\}$. Compute the following quantities:

(a) $\mathbf{E}[\max(X, Y)]$.

(b) $\mathbf{E}[\min(X, Y)]$. (*Hint:* There is a way to get the answer to this reasonably quickly using the previous part.)

(c) $\mathbf{Var}[X - Y]$.

(d) $\mathbf{Var}[|X - Y|]$.

Problem 7. **[10 points]**

(a) Let X be a random variable whose value is an observation drawn uniformly at random from the set $\{-n, -n+1, \ldots, -2, -1, 0, 1, 2, \ldots, n-1, n\}$. Let $Y = X^2$. Show that X and Y are *not* independent, but that $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$.

(b) Show in general for any two random variables X and Y that if $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$ then $\mathbf{Var}[X + Y] = \mathbf{Var}[X] + \mathbf{Var}[Y]$. In particular, this shows that it is possible for non independent variables to have variances that exhibit linearity.

Problem 8. **[12 points]**

Suppose you flip a fair coin 100 times. The coin flips are all mutually independent. What is an upper bound on the probability that the number of heads is at least 70...

(a) ...according to Markov's Inequality?

(b) ...according to Chebyshev's Inequality?

(c) ...according to Chernoff's Bound?