Quiz 2 Solutions

Problem 1. (10 points)

Give tight (Θ) bounds for

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}}$$

You do not need to prove your answer formally, but you should show how you derived it.

We use the integration method for bounding summations.

$$\begin{array}{rll}
\int_1^n \frac{1}{\sqrt{x}}dx & \leq \sum_{i=1}^{n} \frac{1}{\sqrt{i}} & \leq 1 + \int_1^n \frac{1}{\sqrt{x}}dx \\
2x^{\frac{1}{2}}|_1^n & \leq \sum_{i=1}^{n} \frac{1}{\sqrt{i}} & \leq 1 + 2x^{\frac{1}{2}}|_1^n \\
2\sqrt{n} - 2 & \leq \sum_{i=1}^{n} \frac{1}{\sqrt{i}} & \leq 2\sqrt{n} - 1 \\
\lim_{n\to\infty} \frac{2\sqrt{n}-2}{\sqrt{n}} & \leq \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \frac{1}{\sqrt{i}}}{\sqrt{n}} & \leq 1 + \lim_{n\to\infty} \frac{2\sqrt{n}-1}{\sqrt{n}} \\
2 & \leq \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \frac{1}{\sqrt{i}}}{\sqrt{n}} & \leq 2
\end{array}$$

That is

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} = \Theta(\sqrt{n}).$$

Problem 2. (8 points)

For each of the following pairs of functions $f(n)$ and $g(n)$, fill in the blank with O, Ω, Θ, o or ω, depending on whether $f(n) = O(g(n))$, $f(n) = \Omega(g(n))$, $f(n) = \Theta(g(n))$, $f(n) = o(g(n))$ or $f(n) = \omega(g(n))$. If there is more than one relation between $f(n)$ and $g(n)$, write only the strongest one (a relation X is stronger than another relation Y if X implies Y but the converse does not hold). The first line is an example solution. Note: $\lg n = \log_2 n$.

$$
\begin{aligned}
2n &= \underline{\quad\Theta\quad}\ (n) \\
\lg(n^3) &= \underline{\quad\Theta\quad}\ (\lg n) \\
n! &= \underline{\quad\omega\quad}\ (4^n) \\
2^{\log_3 n} &= \underline{\quad o\quad}\ (n^2) \\
\textstyle\sum_{i=1}^{n} 1/i &= \underline{\quad\omega\quad}\ (1) \\
n^{4.01} &= \underline{\quad\omega\quad}\ (n^4 \lg n) \\
\textstyle\sum_{i=n}^{\infty} 1/i^2 &= \underline{\quad o\quad}\ (1) \\
\tbinom{n}{10} &= \underline{\quad\Theta\quad}\ (n^{10}) \\
8^{\lg n} &= \underline{\quad\Theta\quad}\ (n^3)
\end{aligned}
$$

Problem 3. (12 points)
Find asymptotic bounds for the following recurrences. You may assume that $T(n) = \Theta(1)$ for $n \leq 1$ in each part.

(a) $T(n) = 2T(n/2) + \Theta(n)$

$2(\frac{1}{2})^p = 1 \Rightarrow p = 1$. Hence we have that

$$T(n) = \Theta(x(1 + \int_1^x \frac{u\,du}{u^2})) = \Theta(x(1 + \ln x)) = \Theta(x \ln x)$$

(b) $T(n) = 4T(n/2 + \sqrt{n}) + \Theta(n^2)$

$4(\frac{1}{2})^p = 1 \Rightarrow p = 2$. Hence we have that

$$T(n) = \Theta(x^2(1 + \int_1^x \frac{u^2\,du}{u^3})) = \Theta(x^2(1 + \ln x)) = \Theta(x^2 \ln x)$$

(c) $T(n) = T(\lceil n/3 \rceil) + 2T(\lfloor n/4 \rfloor) + \Theta(n)$

We rewrite the recurrence as $T(n) = T(n/3 + (\lceil n/3 \rceil - n/3)) + 2T(n/4 + (\lfloor n/4 \rfloor - n/4)) + \Theta(n)$. By taking logs on both sides of $(\frac{1}{3})^p + 2(\frac{1}{4})^p = 1$ and simplifying, we get $p = \frac{\ln 2}{\ln 12} < 1$. Hence we have that

$$T(n) = \Theta(x^p(1 + \int_1^x \frac{u\,du}{u^{p+1}})) = \Theta(x^p(1 + x^{1-p}) = \Theta(x)$$

Problem 4. (10 points)
Unbeknownst to many, Leonardo Fibonacci had a brother Eduardo who invented the following sequence of numbers that he called *Eduardo* numbers:

$$e_0 = 1$$
$$e_1 = 3$$
$$\forall i \geq 2 : e_i = e_{i-1} + e_{i-2}$$

Eduardo (who was not so good at math) made 2 conjectures about his numbers:

$$C1 : \forall i \geq 0 : e_i = f_{i+1} + 2f_i$$
$$C2 : \forall i \geq 0 : e_i = f_{i+2} - f_i$$

where f_i is the ith Fibonacci number ($f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$).

Decide whether or not Eduardo's conjectures are true. Prove your answers. (Hint: do not repeat Eduardo's mistake by trying to solve the recurrence exactly – there is a much faster way to determine if each conjecture is true.)

The first conjecture can be proven true by induction as follows:

Base cases: $e_o = 1 = f_1 + 2f_0, \; e_1 = 3 = 1 + 2 = f_2 + 2f_1$

Induction step: Suppose $e_i = f_{i+1} + 2f_i$ for $i < k$, where $k \geq 2$. We then have that

$$e_k = e_{k-1} + e_{k-2} = f_k + 2f_{k-1} + f_{k-1} + 2f_{k-2} = f_{k+1} + 2f_k$$

The second conjecture is false. A counterexample is $e_1 = 3 \neq f_3 - f_1 = 1$.

Problem 5. (10 points)

In the following problem there is no direct connection between the two parts.

(a) Suppose that each pair of a genetically engineered species of rabbits left on an island produces two new pairs of rabbits at the age of one month and six new pairs of rabbits at the age of two months and every month afterwards. None of the rabbits ever die or leave the island. Find a recurrence relation for the number of *pairs* of rabbits on the island n months after one newborn pair is left on the island. Do not solve the recurrence relation.

Let $N(t)$ be the number of pairs of rabbits after t months. Then $N(0) = 1$, $N(1) = 3$, and we have that, for $t \geq 2$:

$$N(t) = 7N(t-2) + 3(N(t-1) - N(t-2)) = 3N(t-1) + 4N(t-2)$$

(b) Solve the following inhomogeneous recurrence relation:

$$f(n) = 6f(n-1) - 9f(n-2) + 2^n, n \geq 2$$

The initial conditions are $f(0) = 5, f(1) = 26$.

The characteristic equation $r^2 - 6r + 9$ has a repeated root $\alpha = 3$. Hence the homogeneous solution has the form $f_h(n) = A3^n + Bn3^n$, for some constants A and B. To find a particular solution, we try $f_p(n) = K2^n$. Plugging into the difference equation gives:

$K2^n = 6K2^{n-1} - 9K2^{n-2} + 2^n \Rightarrow K = \frac{6K}{2} - \frac{9K}{4} + 1$
$\Rightarrow 4K = 12K - 9K + 4 \Rightarrow K = 4$

Hence the complete solution has the form $f(n) = A3^n + Bn3^n + 2^{n+2}$. We now use the initial conditions to solve for A and B:

$f(0) = 5 = A + 4 \Rightarrow A = 1$
$f(10 = 26 = 3 + 3B + 8 \Rightarrow B = 5$

Hence the solution is $f(n) = 3^n + 5n3^n + 2^n, n \geq 2$.

Problem 6. (10 points)

(a) Prove, by direct calculation, that the following binomial identity is true for all l, m, n such that $0 \leq l \leq m \leq n$:

$$\binom{n}{m}\binom{m}{l} = \binom{n}{l}\binom{n-l}{m-l}$$

We just expand the binomial coefficients and cancel:

$$\binom{n}{m}\binom{n}{l} = \frac{n!}{m!(n-m)!}\frac{m!}{l!(m-l)!} = \frac{n!}{(n-m)!}\frac{1}{l!(m-l)!} = \frac{n!}{l!(n-l)!}\frac{(n-l)!}{(m-l)!(n-m)!} = \binom{n}{l}\binom{n-l}{m-l}$$

(b) Now prove the same identity combinatorially. Let A be an n-element set. Explain how the left-hand and right-hand sides of the identity in part a) may be interpreted as the number of ways of choosing two subsets B and C of A, where $|B| = m$, $|C| = l$, and $C \subseteq B$.

The left hand side corresponds to first choosing B, then C. The number of ways of doing this is, using the product rule, $\binom{n}{m}\binom{n}{l}$. The right hand side corresponds to first choosing C, and then choosing $B - C$. The number of ways of doing this is, using the product rule, $\binom{n}{l}\binom{n-l}{m-l}$. Since both sides are equivalent to the number of ways of choosing subsets B and C such that $|B| = m$, $|C| = l$, and $C \subseteq B$, the identity holds.

Problem 7. (30 points)

Four distinct children approach you on halloween, requesting candy. You have ten pieces of candy to distribute among them. In how many different ways can you distribute the ten pieces if: (You must specify your answer as an integer, or as a ratio of products of integers, or as an integer raised to the power of an integer)

(a) All the pieces are the same (Reese's Peanut-butter Cups$^{(TM)}$). No other restrictions.

$$\binom{10+4-1}{10} = \binom{13}{10} = \frac{13!}{3!10!}$$

(b) All the pieces are the same, and each child must receive at least one piece. No other restrictions.

$$\binom{6+4-1}{6} = \binom{9}{6} = \frac{9!}{3!6!}$$

(c) All the ten pieces are different. No other restrictions.

$$4^{10}$$

(d) Five pieces are Reese's Peanut-Butter Cups $^{(TM)}$ and the other five are Nestle's Crunch $^{(TM)}$. No other restrictions.

$$\binom{5+4-1}{5}^2 = \binom{8}{5}^2$$

(e) All the pieces are the same. No child gets more than 4 pieces. No other restrictions.
We use the formula for 10-combinations of 4 elements with repetition with each element occuring fewer than 5 times, and get:

$$\binom{13}{3} - \binom{4}{1}\binom{8}{3} + \binom{4}{2}\binom{3}{3} = 68$$

Problem 8. (10 points)
In the following problem, you can leave your answers in closed form, that is, you do not have to write down the coefficients of the generating functions explicitly.

(a) Give a generating function for the number of ways to compose a bag of chocolate donuts, which contains at least 3 donuts.

$$G(x) = \frac{x^3}{1-x}$$

(b) Give a generating function for the number of ways to compose a bag of glazed donuts, which contains at most 4 donuts.

$$G(x) = \frac{1-x^5}{1-x}$$

(c) Give a generating function for the number of ways to compose a bag of coconut donuts, which contains exactly 2 donuts.

$$G(x) = x^2$$

(d) Give a generating function for the number of ways to compose a bag of plain donuts, where the number of donuts in the bag is a multiple of 4.

$$G(x) = \frac{1}{1-x^4}$$

(e) Give a generating function for the number of ways to compose a bag of donuts where the donuts are of 4 kinds (chocolate, glazed, coconut and plain) and all the following conditions hold:

- at least 3 are chocolate donuts.
- at most 4 donuts are glazed.
- there are 2 coconut donuts.
- the number of plain donuts is a multiple of 4.

$$G(x) = \frac{x^3}{1-x}\frac{1-x^5}{1-x}x^2\frac{1}{1-x^4} = \frac{x^5(1+x^2+x^3+x^4)}{(1-x)(1-x^4)}$$