Problem Set 9 Solutions

Problem 1. **(a)** We require each $x_i \geq 2$. This uses up 12 of the 29 total required, so the problem is the same as finding the number of solutions to $x'_1 + x'_2 + x'_3 + x'_4 + x'_5 + x'_6 = 17$ with each x'_i a nonnegative integer. The number of solutions is therefore $\binom{6+17-1}{17} = \binom{22}{17} = 26,334$.

(b) The restrictions use up 22 of the total, leaving a free total of 7. Therefore the answer is $\binom{6+7-1}{7} = \binom{12}{7} = 792$.

(c) The number of solutions without restriction is $\binom{6+29-1}{29} = \binom{34}{29} = 278,256$. The number of solutions violating the restriction by having $x_1 \geq 6$ is $\binom{6+23-1}{23} = \binom{28}{23} = 98,280$. Therefore the answer is $278256 - 98280 = 179,976$.

(d) The number of solutions with $x_2 \geq 9$ (as required) but without the restriction on x_1 is $\binom{6+20-1}{20} = \binom{25}{20} = 53130$. The number of solutions violating the additional restriction by having $x_1 \geq 8$ is $\binom{6+12-1}{12} = \binom{17}{12} = 6188$. Therefore the answer is $53130 - 6188 = 46,942$.

Problem 2. By Theorem 3 the answer is $11!/(4!4!2!) = 34,650$.

Problem 3. We need to calculate separetely, using Theorem 3, the number of strings of length 5, 6, and 7. There are $7!/(3!3!1!) = 140$ strings of lenght 7. For strings of length 6, we can omit the R and form $6!/(3!3!) = 20$ strings; omit an E and form $6!/(3!2!1!) = 60$ strings, or omit an S and also form 60 strings. This gives a total of 140 strings of length 6. For strings of length 5, we can omit two E's or two S's, each giving $5!/(3!1!1!) = 20$ strings; we can omit one E and one S ($5!/(2!2!1!) = 30$ strings); or we can omit the R and either an E or an S ($5!/(3!2!) = 10$ strings each). This gives a total of 90 strings of length 5, for a grand total of 370 strings of length 5 or greater.

Problem 4. We follow the hint. There are 5 bars (chosen books), and therefore there are 6 places where the 7 stars (nonchosen books) can fit (before the first bar, between the first and second bars,..., after the fifth bar). Each of the second through fifth of these slots must have at least one star in it, so that adjacent books are not chosen. Once we have placed these 4 stars, there are 3 stars left to be placed in 6 slots. The number of ways to do this is therefore $\binom{6+3-1}{3} = \binom{8}{3} = 56$.

Problem 5. Each term must be of the form $Cx_1^{n_1}x_2^{n_2}\ldots x_m^{n_m}$, where the n_i's are nonnegative integers whose sum is n. The number of ways to specify a term, then, is the number of nonnegative integer solutions to $n_1+n_2+\ldots+n_m=n$, which by Theorem 2 is $\binom{m+n-1}{n}$. Note that the coefficients C for these terms can be computed using Theorem 3.

Problem 6. **(a)** $1/(1-x)$.

(b) $1+x+x^2$.

(c) $x^2/(1-x)$.

(d) $x^3/(1-x)$.

(e) $x^3+x^4+2x^5+x^6/(1-x)=\frac{x^3}{1-x}+x^5$.

(f) $\sum_n x^{3n}=1/(1-x^3)$.

(g) $(1+x)/(1-x)^3$.

(h) $2/(1-x)^3$.

(i) $\sum_n n^2(n+2)x^n=\sum_n n^3x^n+2\sum_n n^2x^n=\frac{x+4x^2+x^3}{(1-x)^4}+2\frac{x+x^2}{(1-x)^3}=\frac{3x+4x^2-x^3}{(1-x)^4}$.

(j) $1/(1-3x)+1/(1-5x)+1/(1-7x)$.

(k) e^x.

(l) $e^x/(1-x)$.

Problem 7. **(a)** $\frac{8x}{4x^2-1}=\frac{8x}{(2x-1)(2x+1)}=\frac{2}{2x-1}+\frac{2}{2x+1}$.

(b) $x^2-4x+4+\frac{5}{x+2}-\frac{3}{x+1}$.

(c) $\frac{x-5}{x^3+x^2-x-1}=\frac{x-5}{(x-1)(x+1)^2}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$

Cover-up method gives: $A=-1,\ C=3 \implies x-5=-(x+1)^2+B(x-1)(x+1)+3(x-1) \implies x-5=-x^2-2x-1+Bx^2-B+3x-3 \implies \quad B=1$.

Hence, we have:

$\frac{x-5}{x^3+x^2-x-1}=\frac{-1}{x-1}+\frac{1}{x+1}+\frac{3}{(x+1)^2}$.

Problem 8. Let $A(x)=\sum_{i=0}^{\infty}a_ix^i$. Then, we get from the recurrence that $A(x)=2x^2A(x)+x^3A(x)+1+2x+2x^2$, since $a_0=2^0=1,\ a_1=2^1=2,\ a_2=2^2=4$. Solving for $A(x)$ we get

$$A(x)=(2x^2+2x+1)/(1-2x^2-2x^3)=(-2x^2-2x-1)/(x+1)(x^2+x-1)$$

Roots of x^2+x-1 are $\alpha_+=(-1+\sqrt{5})/2$ and $\alpha_-=(-1-\sqrt{5})/2$.

Hence, we have:

$$A(x) = \frac{1}{1+x}\cdot\frac{-1}{(-1-\alpha_+)(-1-\alpha_-)}+\frac{1}{x-\alpha_+}\cdot\frac{-2\alpha_+^2-2\alpha_+-1}{(\alpha_++1)(\alpha_+-\alpha_-)}+\frac{1}{x-\alpha_-}\cdot\frac{-2\alpha_-^2-2\alpha_--1}{(\alpha_-+1)(\alpha_--\alpha_+)}$$

Some identities of α_+ and α_-:
$1+\alpha_+ = -\alpha_-,\quad 1+\alpha_- = -\alpha_+,\quad \alpha_+ - \alpha_- = \sqrt{5},$
$\alpha_*^2 + \alpha_*^2 - 1 = 0,$ where $*$ stands for either $+$ or $-$, $\alpha_+\alpha_- = -1.$
Employing these identities, we can write:

$$\begin{aligned}
A(x) &= \frac{1}{1+x}(\frac{-1}{-\alpha_+\alpha_-}) + \frac{1}{x-\alpha_+}(\frac{-3}{(-\alpha_-)\sqrt{5}}) + \frac{1}{x-\alpha_-}(\frac{-3}{(-\alpha_+)\sqrt{5}}) \\
&= \frac{1}{1+x} + \frac{3}{(\alpha_-)\sqrt{5}}\cdot\frac{1}{x-\alpha_+} + \frac{3}{(\alpha_+)\sqrt{5}}\cdot\frac{1}{x-\alpha_-} \\
&= \frac{1}{1+x} + \frac{3}{\sqrt{5}}\cdot\frac{1}{1-\alpha_- x} + \frac{3}{\sqrt{5}}\cdot\frac{1}{1-\alpha_+ x}
\end{aligned}$$

This implies that

$$\begin{aligned}
a_n &= (-1)^n + \frac{3}{\sqrt{5}}(-\alpha_-)^n + \frac{3}{\sqrt{5}}(-\alpha_+)^n \\
&= (-1)^n + 3(\frac{1}{\sqrt{5}}(\alpha_+)^n + \frac{1}{\sqrt{5}}(\alpha_-)^n) \\
&= (-1)^n + 3F(n) \qquad \text{where } F(n) \text{ is the } n^{th} \text{ Fibonacci number}
\end{aligned}$$

We now verify by induction that indeed $a_n = (-1)^n + 3F(n)$.

Let $P(n)$ be the statement that "For $k \le n,\ a_n = (-1)^n + 3F(n)$". Then we have:

Proof.
Base cases:
$P(0) : a_0 = 1 + 3F(0) = 1 + 0 = 1$
$P(1) : a_1 = -1 + 3F(1) = -1 + 3 = 2$
$P(2) : a_2 = 1 + 3F(2) = 1 + 3 = 4$
Inductive step. Assume $P(n)$ is true. Then

$$\begin{aligned}
a_{n+1} &= 2a_{n-1} + a_{n-2} \qquad \text{by definition.} \\
&= 2\left((-1)^{n-1} + 3F(n-1)\right) + (-1)^{n-2} + 3F(n-2) \qquad \text{by inductive hypothesis} \\
&= (-2+1)(-1)^{n-2} + 3\left(2F(n-1) + F(n-2)\right) \\
&= -(-1)^{n-2} + 3\left(F(n-1) + F(n-1) + F(n-2)\right) \\
&= (-1)^{n-1} + 3\left(F(n-1) + F(n)\right) \\
&= (-1)^{n+1} + 3F(n+1)
\end{aligned}$$

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Problem 9. **(a)** GF for $\langle a_k \rangle = \frac{x}{1-x^2} \implies A(x) = (\frac{x}{1-x^2})^4$.

(b) GF for $\langle a_k \rangle = \frac{1}{1-x^3} \implies A(x) = (\frac{1}{1-x^3})^4$.

(c) GF for: $\langle a_1 \rangle = 1,\ \langle a_2 \rangle = 1+x,\ \langle a_3 \rangle, \langle a_4 \rangle = \frac{1}{1-x} \implies A(x) = \frac{1+x}{(1-x)^2}$.

(d) GF for: $\langle a_1 \rangle = x + x^3 + x^{11},\ \langle a_2 \rangle = x^2 + x^4 + x^5,\ \langle a_3 \rangle, \langle a_4 \rangle = \frac{1}{1-x} \implies$ $A(x) = \frac{(x+x^3+x^{11})(x^2+x^4+x^5)}{(1-x)^2}$.

(e) GF for $\langle a_k \rangle = \frac{x^{10}}{1-x} \implies A(x) = (\frac{x^{10}}{1-x})^4$.